STIMULATING CHILDREN'S UNDERSTANDING OF RECTANGULAR SOLIDS
MADE OF SMALL CUBES

by

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ABSTRACT

The purpose of this study was to investigate fourth grade students' understanding of rectangular solids made of small cubes as they engaged in enumeration tasks and to stimulate their understanding through equal sharing activities involving cube buildings.

In order to achieve this purpose, a quasi-experimental design was utilized with four 4th graders. In the first phase, clinical interviews were conducted individually to assess their level of functioning both in concrete and pictorial situations while engaged in cube enumeration tasks. In the second phase, participants were engaged in equal sharing of spatial constructions using drawings, concrete buildings, loose cubes, and colored pens as materials. In the last phase, post-clinical interviews were conducted to probe their improvements as revealed by their use of enumeration strategies.

Results showed that students used three distinct conceptualizations for the arrays of cubes depending on what they formed as a unit and how they structured the whole building. Initially, their structuring was distracted by the complexity of the buildings and none of them used the same strategies consistently across the problems. During the instruction, they exhibited the same three conceptualizations and their transitioning from one to the other, benefiting from mainly three sources of interaction: peer, instructor, and materials. After the instruction, all the students consistently used layering strategies regardless of the complexity of the buildings. Equal sharing situations paved the road in establishing units, composite units, and unit iteration.
To my wife, Emine,

without whose love and support

this accomplishment

could never have been achieved.
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CHAPTER 1

INTRODUCTION

In many areas of mathematics, it is of great value to be able to visualize and represent 3D configurations and to comprehend the geometrical relations among the various parts of a figure (Mitchelmore, 1980; Wilson & Osborne, 1992). Mathematics is the activity of constructing patterns and relationships (Wheatley, 1997; Zimmerman & Cunningham, 1991) in any form such as numerical, symbolical, and figural. Finding patterns (invariants) in the relationships among and between elements of figures is in the domain of geometry (Goldenberg, Cuoco, and Mark, 1998).

There is a two-way interaction between geometrical and numerical reasoning. For example, it is historically evident that geometry has played an important role in the development of mathematics (Goldenberg et al., 1998). Similarly, visual presentations can be a powerful introduction to the complex abstractions of mathematics (Battista, 1994; Battista & Clements, 1998, 1996; Bishop, 1989; Clements, Battista, Sarama, and Swaminathan, 1997). In fact, many concepts and processes in school mathematics can be tied to visual interpretations. For this purpose, visual models can be built that reflect (a large part of) the underlying mathematical structure (Eisenberg & Dreyfus, 1989) to make it more meaningful and accessible for students.

Barwise & Etchemendy (1991) further claimed that visual forms of representation can be important, not just as heuristic and pedagogic tools, but as legitimate elements of mathematical proofs. Still, however, many people do not consider visual solutions as finished work in mathematics. As they stand now, visual representations serve as an
intuional proof to stimulate further thought because they help people gain insights into calculations in mathematics (Goldenberg et al., 1998). Piaget and Inhelder (1967) believed that intuition was the instrument of invention, whereas demonstration or geometrical reasoning in the strict sense was a matter of logical analysis. And indeed, by developing axiomatic procedures, modern geometry has attempted to segregate the two processes as completely as possible. The radical separation of intuition from logic or axiomatics has never been achieved in practice; and in fact, is unattainable in principle (Piaget & Inhelder, 1967). In other words, in mathematical reasoning there always remains some links with intuitive experience with which visual clues are used to stimulate further thought (Nelsen, 1993) since visualization enhances a global and intuitive view and understanding (Hershkowitz, 1989).

There has been a growing awareness that mathematics is primarily concerned with spatial, geometrical, or configurational concepts (Smith, 1964). This awareness has led an increasing interest in visual models as instructional aids for teaching mathematics at all levels. For more than 100 years, mathematics educators have been interested in the visual and figural representations of mathematical ideas both in the work of individuals and in the process of teaching about these ideas (Bishop, 1989).

Cubes and cube configurations such as rectangular arrays of cubes are among the instructional aids used in school mathematics. For example, drawings of rectangular solids made of small cubes have been included in textbooks to introduce the students to the concept of volume (Ben-Chaim, Lappan, and Houang, 1985). They have been also included in large-scale educational assessment studies to measure students' knowledge of
the volume concept (Hirstein, 1981). Students are required to find the number of cubes contained in rectangular buildings. Battista and Clements (1998) argued that "the reasoning required to complete such tasks is important because it builds the cognitive framework for understanding the measurement of volume and the formulas for determining the volume" (p. 258). Indeed, such geometrical representations as differently piled rectangular solids are used to contribute to students' understanding of number and measurement ideas such as length, area, and volume (Geddes & Fortunato, 1993). However, the development of students' understanding and their strategies for meaningfully enumerating cubes in rectangular solids is far more difficult than has previously been believed (Battista & Clements, 1998).

According to the results of the second National Assessment of Educational Progress, fewer than 40% of 17 year-olds could solve the problems related to the volumes of rectangular solids made of small cubes (Hirstein, 1981). The types of errors being made were related to either the number of visible cube faces or the number of visible cubes. Hirstein concluded that many of the errors made by students were due in part to their confusion between volume and surface area. Ben-Chaim et al., (1985) found almost the same patterns of low performance with approximately 1,000 students from fifth through eighth grades. They additionally found that some of the students double counted the cubes along the edges. Since the researchers presented students with only pictorial representations, they concluded "students of given age had difficulty relating isometric type drawings to the rectangular solids they represent" (p. 389).
The research by Battista and Clements (1998, 1996) revealed that it was neither a matter of confusion between volume and surface area nor a lack of understanding pictorial representations of rectangular buildings but a matter of improper spatial structuring. Battista and Clements (1996) defined spatial structuring as "the mental act of constructing an organization or form for an object or set of objects" (p. 282). The process of spatial structuring includes establishing units, establishing relationships between units (such as how they are placed in relation to one another), and recognizing that a subset of the objects, if repeated properly, can generate the whole set - the repeating subset forming a composite unit (Battista & Clements, 1996). In other words, forming units based on three dimensions such as unit cubes and units of units such as columns and layers, and iterating them along the third dimension are necessary mental constructions for properly structuring rectangular arrays of cubes. How students structured the arrays of cubes can be inferred from their strategies they use while dealing with them.

Battista and Clements (1998, 1996) described student strategies and categorized them into five main groups. First was the misapplication of volume formula. Students in this group multiplied some numbers irrelevant to the dimensions. Students in the second group used the formula rotely without any indication of the layer-structure of the buildings. For example, they used the formula but they could not give any justification for why it worked. Students in the third group conceptualized the set of cubes in terms of its faces and they accounted only for the visible or outside cubes. The fourth group of students conceptualized the set of cubes as space filling but did not yet utilize layers. Initially, they counted cubes unsystematically and then started to organize them in some
local groups. Students of the last group conceptualized the set of cubes as forming a rectangular array organized into layers. They started counting sub units of layers. They continued with additive and multiplicative iterations of layers.

Layering is considered the most efficient strategy for proper structuring of the arrays of cubes. The students with this conceptualization are the most likely, with understanding, to produce correct answers. Also, layering strategy very closely corresponds to the volume formula in terms of the logical operations behind it. Seeing the solids in terms of iterable units such as layers is a valuable skill that might be useful in later mathematical learning. For example, a more sophisticated application of layering strategies can be seen in calculus while reasoning about the volumes of revolutions. This seems a gradual development, however. An individual does not "read off" a structure from objects, but instead creates a structure as a result of his or her mental actions concerning the objects (Battista & Clements, 1996; Cobb, Yackel, & Wood 1992).

An individual's mental actions are influenced not only by available conceptual structures but also the intentions and the ongoing social interaction in which s/he is involved (Wheatley & Cobb, 1990). In other words, The ways in which children think about spatial structures are influenced not only by the ways in which their own knowledge is structured, but also by the ways in which the context for thinking about and discussing these entities is structured. For example, equal sharing is a socially based, semantically rich context (Empson, 1995) in which students' understanding about equal amounts in spatial constructions can be facilitated.
Children have intuitive knowledge of equal sharing because of their experiences. This knowledge can be facilitated in similar contexts such as through equal sharing of spatial constructions. Constructing buildings by giving equal shares may also provide a similar context in which thinking about and discussing spatial and numerical entities can be structured meaningfully. Through these activities students can construct rectangular buildings while they deconstruct them through equal sharing. That way, the structural elements of cube configurations can be pinpointed through equal amounts of contribution and sharing activities. It is important where the students' attention is directed during the instruction (Anderson, Reder, & Simon, 1995).

Numerical reasoning also plays an important role in this endeavor (Battista & Clements, 1996). In fact, there is a synergistic effect between numerical reasoning and learning geometry (Clements et al., 1997; Lubinski & Humpreys, 1990) such that one provides input for, or constrains, the other (Battista & Clements, 1996). Children attach meaning to the numbers when they learn one-to-one correspondence of numbers with the objects. Before that, numbers are nothing more than just words memorized and used, for example, in singing. In addition, children's initial understanding of geometry is mainly visual (van Hiele, 1986). In fact, "mathematical understanding is constructed to a large extent in images, many of which are spatial in nature" (Clements et al., 1997).

From the forgoing discussions, it seems that finding the number of cubes in rectangular buildings is important for developing valuable mathematical skills; however, students have difficulty with it. The major source of the difficulties is related to their use of inadequate strategies depending on their cognitive constructions. This study assumes
that learning arises as an independent contribution of the interacting student (Steffe, Thompson, & Glasersfeld, 1998). Instructional tasks, materials, and others (students and teacher) are also influential components of learning environments. For example, the experience children gain through interactions with adults greatly influences their construction of mathematical knowledge (Cobb & Steffe, 1983). Instructional interactions in general enable children to gain access to resources and tools that both reorganize and transform their thinking, resulting in more powerful cognitive structures. While solving problems using their prior knowledge (some of them are informal/intuitive knowledge), children discover new ways of looking at things, leading the development of that knowledge into more sophisticated knowledge.

Developing meaningful activities that help students transition to higher levels of thinking is an important task for educators. The fact that there are both numerical and spatial aspects of the problem of finding the number of cubes in rectangular buildings is an advantage because there is a synergistic effect between numerical reasoning and geometry learning (Clements et al., 1997). Therefore, it appears that if we provide students with appropriate activities that emphasize both numerical and spatial reasoning in a socially desirable context we can affect their spatial structuring so that they can move towards using more viable strategies in a mathematically meaningful way.

Purpose

Previous research has described students' solution strategies and errors in dealing with three-dimensional cube arrays and has provided several cognitive constructions and operations that students require for enumerating cubes in such arrays. However, it has
not explicated how the instruction that takes both the numerical and spatial-structural elements of cube configurations into consideration and that is given in a semantically rich social context would affect the students' spatial structuring of cube arrays.

Specifically, this study is intended to reveal whether the activities provided in the context of equal sharing that emphasize both numerical and spatial aspects of cube configurations cause any improvement in students' strategies leading them to use composite units and unit iteration in enumerating the cubes contained in rectangular buildings.

Significance

First the current study showed whether these kinds of activities would cause any improvement in students' spatial structuring as revealed by their use of strategies. Second, the study also highlighted how students used numerical and spatial information interchangeably. Since the activities stressed both numerical and spatial aspects of configurations it might be possible to see what kind of information students relied on first, or more, and if there were any change in this reliance. This study also used a different paradigm to formulate the activities by embedding them in a socially desirable context, equal sharing of three-dimensional spatial entities. Third, this study revealed how this context as a whole would affect students' approach to rectangular solids made of small cubes.

The information gained in this study is crucial for three reasons. First, it provides feedback for the development of effective instructional materials. Second, it sheds light on children's mathematical thinking in an area that intersects with both numerical and
spatial domains. Third, it also highlights some part of an often-neglected area in mathematics, visualization in three-dimensional geometry (Gutierrez, 1995).
CHAPTER 2
REVIEW OF LITERATURE

In the first chapter, four topics are included. They are as follows: the role of visual models in stimulating mathematical thought, the importance of finding the number of cubes in rectangular buildings, the existence of students' difficulties, and the necessity and the ways of helping students to further their mathematical thinking.

In this chapter, the literature is reviewed in order to clarify concepts and relevant theories that might help the reader understand students' spatial thinking in general and spatial structuring of three-dimensional cube arrays in particular. Therefore, the topics of discussion in this chapter are as follows: (a) visualization in mathematics; (b) development of three-dimensionality in children; (c) volume, spatial structuring, and student strategies; and (d) stimulation of spatial structuring. The chapter concludes with a summary and a list of the research questions formulated for this study.

Visualization in Mathematics

Because of the disagreement among researchers and educators on the terminology used and on what constitutes visualization, it is difficult even where to start and how to start defining visualization. For the purposes of this study visualization from the psychology and mathematics education perspective is clarified.

In psychometric research, the term spatial visualization, rather than visualization alone, is used as one of the subfactors of spatial ability, which is considered a factor in human intelligence. McGee (1979) succinctly defined spatial visualization as the ability to mentally rotate, manipulate, and twist two- and three-dimensional objects presented
pictorially. However, the tasks included in spatial visualization tests demand some other skills and operations. For example, some tasks demand the ability to imagine movement or internal displacement among the parts of a total configuration and comprise multi-step reasoning about parts of objects with multiple solution paths (Friedman, 1995). It may also involve many distinct steps in transforming an item or items, sometimes involving movement of only parts of the items. Transformations from 2D to 3D and vice versa through folding, unfolding (Pellegrino, Alderton, & Shute, 1984) and/or mental integration are considered exercising spatial visualization (Ben-Chaim et al., 1985).

Another subfactor in spatial ability, less often separated, is tied to the speed of comparatively simple mental judgements about rotations of objects (Cronbach, 1984). Some call it mental rotation (McGee, 1979), others label it as spatial relations (Pellegrino, et al., 1984). Basically, it involves mental rotations of rigid objects as a whole, either two or three-dimensional. In fact, mental rotation tasks, especially those in three dimensions, are the best test of "pure" spatial reasoning, precisely because they are the most resistant to solutions by sequential processing (Friedman, 1995).

Ordinarily, spatial visualization tasks can be broken down into a sequence of steps involving simple mental rotation tasks (Friedman, 1995) either two- or three-dimensional. Some additional aspects of spatial visualization make it even more important in many scientific and academic disciplines including mathematics (Zimmerman & Cunningham, 1991). In a meta-analysis, Friedman (1995) found that when space-math correlation(s) were grouped by spatial skill and averaged over studies, they ranged from about 0.30 for mathematics and two-dimensional rotation tests to 0.45 for mathematics and three-
dimensional visualization. Three-dimensional visualization is the best relating spatial factor. Three-dimensional rotation is also highly correlated with both computational ($r=0.38$) and higher-level mathematics ($r=0.43$) for younger students (9-14 year-olds). Correlations were generally higher for females, younger students, and visualization tasks.

Some may find these correlations too low in order to claim a strong relationship between spatial ability and mathematical achievement. And, in fact, they are lower than the correlation between mathematical and other skills such as verbal skills (Friedman, 1995). However, it all depends on what we count as mathematics (Battista, 1994; Wheatley, 1997). According to Battista (1994), the kinship between mathematical and spatial thinking applies to the learning of conceptual domains, not to rote memorization of procedures for manipulating symbols as emphasized in much of current school mathematics. For instance, Wheatley (1997) found that students who used images in their reasoning were more successful in solving non-routine mathematics problems than those who procedurally approached the tasks. In order to further elaborate on this issue one should look from a cognitive perspective.

Cognitive approach to human cognition focuses on how the human memory system acquires, transforms, compacts, elaborates, encodes, retrieves, and uses information (Miller & Burton, 1994). There are two kinds of information to be represented in memory sentential and spatial information. Evidence for memory storage indicates that representations can be both meaning-based and perception-based, that is, propositions and imagery (Miller & Burton, 1994). Spatial information is represented in memory as mental images that preserve the topological and geometrical relations among
the components of the problem (Shepard, 1978). Images are regarded as faint copies of movements and sensations associated with visual stimuli in memory, and obtained through repeated experiences with the external world (Eliot, 1987). Basically, human minds create and act on mental images, which are believed to be quasi-pictorial entities.

Mental images are acquired or created through sensations and held in working (short-term) memory. Attention is required for both the creation and maintenance of mental images. They are then stored in long-term memory. They can be evoked from long term memory through intention and cues from the environment and transformed and/or manipulated while they are in the working memory (Miller & Burton, 1994).

Mental image enables one to reason about or recall the appearance of an object in its absence (Roth & Kosslyn, 1988). There is evidence that mental representations are isomorphic to the actual images (Cooper, 1990; Roth & Kosslyn, 1988; Shepard, 1978). The degree of isomorphism depends on the attention and processing experienced during the perception. According to dual coding theory (Paivio, 1991), mental images are not exact copies of the actual images but instead contain certain information encoded during the interaction with the stimulus image; that is why they can be activated by any of the cues such as appearance, sound, smell, and taste but not in complete detail. The encoded images, alongside verbal vocabularies act upon human cognition, information processing, and the communication processes (Paivio, 1991; Miller & Burton, 1994). The nature of imagery has been shown to be constructive and perceptual (Thompson, 1990).

Mathematics educators tend to use visualization and visual imagery instead of spatial visualization and mental imagery. Even if they use the same terminology they
tend to give them broader meanings. They take the term visualization, for example, to describe the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated (Presmeg, 1997; Zimmerman & Cunningham, 1991). Similarly, Gutierrez (1996) considered visualization in mathematics as "the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties" (p. 9).

An image is a mental construction for a mathematics educator (Wheatley, 1997). However, it is not restricted to be based on physically visible objects. It is also much more than a mental picture (Thompson, 1996). Images may be of several types such as concrete, dynamic, pattern, or even abstract (Presmeg, 1997). For example, a pattern in a numerical set is not inherently visible but someone can make it visible through imagery.

As seen from the previous discussions, psychological approaches restrict the notions of both spatial visualization and mental imagery to forming a mental image from a visible object, including its imagined movements and the action upon it in its absence. In other words, it is a kind of mental manipulation of inherently visible things in their absence, by means of a mental picture. Mathematics educators assume visualization more than forming mental images of physical objects and their movements. First, to them visualization comprises of making things visible either internally (mentally) and/or externally (diagrammatically). Those things that are to be made visible are not just physical objects but also concepts that are not inherently visible. In mathematics, visualization includes the students' ability to draw an appropriate diagram (with paper
and pencil, or in some cases, with a computer) to represent a mathematical concept or problem and to use the diagrams to achieve understanding and as an aid in problem solving (Zimmerman & Cunningham, 1991).

Based on the task similarity, we could say "finding the number of cubes in rectangular buildings made of small cubes" is clearly a spatial visualization task in a psychometric sense. It includes the manipulation of a multi-part, three-dimensional object, which has some parts invisible that need to be made visible in order to complete the task.

From a cognitive perspective, one should engage in creating a mental picture in order to be able to perform actions on it to reason about the geometrical relations of individual cubes in relation to each other and to the whole building. The task includes the creation of a mental image through analogical reasoning between the external and the internal (mental) image and transforming it into other forms and dimensions. How does one generate or create a mental image of a 3D object?

Roth and Kosslyn (1988) claimed that mental images of 3D objects should also be represented in three dimensions. Similarly, Cooper (1990) argued that adult viewers construct mental representations as perspective-like, embodying structural information about integrated 3D objects even when asked to reason about flat, disconnected orthographic projections. This may be because isometric projections present an approximate perspective view of objects and they appear more object-like than their orthographic equivalents. Additionally, mental images of 3D objects are generated in a
near-to-far sequence (Roth & Kosslyn, 1989). That is, nearer surfaces of imaged objects are generated prior to distant surfaces.

From a mathematics education perspective, "finding the number of cubes in rectangular buildings made of small cubes" is clearly a visualization task. The task includes analogical, analytical, and numerical reasoning that might altogether lead to the formation of the concept of the measurement of volume. As mentioned earlier, analogical reasoning is the formation of the mental image similar to the external object through continuous checking between external object and mental (internal) image for the structural similarity. Analytical reasoning comprises of putting these small pieces into relations to form units, composite units, and the whole object respectively. Numerical reasoning is the abstraction of the numerical properties of units and composite units and iterations of them to form the numerical equality of the whole building. For example, one form of numerical reasoning could be that a six-cube layer is equalized to the number "6" and counted as such. The strategy "counting on" is treating a layer as an iterable composite unit (Battista & Clements, 1996).

The three types of reasoning (analogical, analytical, and numerical) have been summarized that could occur in the enumeration of the cubes in rectangular buildings. The next section is an explanation of how children's understanding of three-dimensionality develops through ages.

The Development of 3-Dimensionality in Children

Many people have trouble handling the 3D case visually, even if they find the 2D case very natural in a visual setting (Eisenberg & Dreyfus, 1989). The presentations of
activities for children on solid shapes, using textbooks or worksheets, for example, meets with the well-known problem of representing three-dimensional solids or frames on flat two-dimensional paper (Orton & Frobisher, 1996) which is an essential skill in many areas of mathematics and science. How do we know about their understanding?

Children's achievement in volumetric (three-dimensional) tasks is assessed through three different procedures: They are (a) given building blocks to build 3D geometrical objects, (b) required to read pictorial representations of objects, and/or (c) asked to draw the graphical representations of objects (Wolf, 1988). The challenge in blocks is to stop using just the volume of individual blocks in favor of learning how to construct larger volumes from many blocks. In reading the pictorial representation of a real world object, one comes to realize that the picture represents a space-filling object, an object with volume. The challenge of drawing is to use the 2D-paper plane to represent the depth and volumes of the spatial world.

Only one procedure may not be enough or appropriate to assess the presence or absence of a given type of information in a representational system (Mandler, 1988). For example, a lack of motor coordination in drawing or a child's inability to draw may hide some part of the actual visualizing (Holloway, 1967). Additionally, young children often lack the ability to understand 2D displays of 3D phenomena (Cohen, 1979) since it needs considerable conventionalizing (Bishop, 1979). Therefore, all three procedures should be utilized in order to get a feeling about the level of a student's functioning.

Understanding three-dimensionality depends on the employment of perspective (Piaget et al., 1970). According to Morss (1987), the employment and awareness of
perspectives were gradual, developmental achievements. Wolf (1988) reported "as late as their third year, children use similar, rather than differentiated systems for depicting the features and locations of parts within a larger whole, independent of whether they are given markers and paper or building blocks" (p. 232). In both their block models and their drawings of houses, for example, children simply accorded a block or a mark for each item they wished to represent in a way that recorded rough location and number but nothing about external contour or exterior volume. In other words, their drawings lacked the distinctive 2D, or graphic, strategies for representing volume with space around it. The children did not even bother with the size of the object.

At the same time, a child's block construction is only uni-dimensional (a pile of items) rather than a fully 3D construction. For example, to represent a house the child may use two or three cubes, placing one on top of the other without any indication of central cavity. However, by the age of five, a child learns to distinguish between the two systems of spatial representation (Wolf, 1988). A five-year-old selectively uses 2D or pictorial strategies to depict the volume of a house. The child uses a contour line to distinguish the figure of the house from the surrounding ground of the paper and depicts the several facets of volume in left-to-right arrangement on the page. Another five-year-old constructs the volume of a house making use of a central cavity and construction along three axes to portray the shape of the form in space (Wolf, 1988).

In other words, between the ages of one and five years children gradually construct the distinctive rules of at least these two systems (2D graphical and 3D construction) of spatial representation (Wolf, 1988). In their drawings, five-year-old
children devise still other ways to portray the volume of individual objects, such as coloring them in or enlarging their outside contours. Viewed from an adult perspective, these are primitive strategies. Nevertheless, they are significant because they signal the children’s awareness that when drawing, they must find 2D means to represent the 3D aspects of their spatial experience (Wolf, 1988). By the age of six, children may try to differ a disc from a sphere in their drawings by means of coloring (Wolf, 1988).

There is evidence that even young children can read graphic representations with a fair degree of sophistication (Wolf, 1988). Murphy & Wood (1981) measured the functional use of pictorial information in reconstruction tasks. Children starting from the age of four were able to use the pictorial information in 3D construction tasks. Performance significantly increased with age. Some of the eight-year-olds could do the task without even looking at the picture. They might have used the spatial information they constructed either through previous experiences or from relating the parts to each other or both. Whatever it was, it was clear that an average eight-year-old could associate certain objects with their drawings.

Holloway (1967) argued that to represent an object in perspective by means of a mental image or drawing necessitated an awareness of the point of view and the resultant apparent changes in the shape of the object as seen from that viewpoint. Perspective representation implied conscious coordination between object and subject, the recognition that both object and the subject occupied the same projective space extending beyond the object, including the observer (Piaget & Inhelder, 1967). Seven year-olds are aware of the distinction between their successive views of the object but cannot imagine
or draw the outcome of these changes in viewpoints (Holloway, 1967). It is only when presented with a ready-made perspective drawing that the child is able to accept it as an adequate rendering of the object from a particular viewpoint (Holloway, 1967). Children between the ages of 8.5 and 9 years of age can systematically apply the rules of perspective even in their drawings.

Mitchelmore (1976, 1980) described school children's drawings of space figures into four main developmental stages. Stage 1, plane schematic: drawings show only a single face as if they are two-dimensional or taken orthogonally. Stage 2, solid schematic: drawings show several faces of the solid that may be visible or nonvisible but this may or may not indicate the depth of the figure portrayed. Stage 3, pre-realistic: only visible faces are included in the drawing and depth is apparent in the drawings. Stage 4, realistic: parallel lines are drawn parallel and depth is represented properly (see Figure 1 for cuboid drawings).

Mitchelmore (1980) assessed children from third, fifth, seventh, and ninth grades to see if the developmental stages were valid for a different culture. Since there was little variation among the first graders in a previous study, he did not include first graders in this study. He found very similar results. Third graders' mean was between the first and second stage. Fifth graders' mean was between Stage 2 and Stage 3A. Seventh and ninth graders' means were not very much different and between Stage 3A and Stage 3B. He concluded that the scores changed in a linear fashion and measured "something that can be justifiably considered as general representational ability" (p. 90).
What is the relation, then, between a child’s representational ability and level of spatial-perceptual development? It seems as if, at any point in time, a child has available one or more schemata (two-dimensional designs) for representing a particular class of space figures (Mitchelmore, 1980). Perception of a simple geometrical figure as representing a solid may be a function of two factors: the ability to perceive pictorial depth of the elements of a figure, and the ability to construct a correct mental representation of the entire figure (Deregowski & Bentley, 1987). According to Mitchelmore (1980), the reason why older children’s schemata appear more realistic than younger children’s is that they are based in a more refined space (Euclidean as opposed to topological). For example, there is little variance in drawing stages at the first grade. Growth of representational ability should be expected to be highly dependent on spatial-perceptual development but always to lag behind it. The lag may be as much as three stages at some ages (Mitchelmore, 1980).
According to Mitchelmore (1980), there are also marked differences in the difficulty of drawing different space figures. The stage of development of a child’s schema for a given space figure depends not only on his or her general level of perceptual development but also on the particular representational problems presented at each stage. For example, young children find the cylinder the most difficult space figure to represent satisfactorily by a simple closed plane figure (possibly because of the curved face).

Considerable differences exist between the subjects drawn from different cultures (and even within cultures) and between subjects differing in such attributes as age and education (Deregowski & Bentley, 1987; Mitchelmore, 1980). In sum, everyone draws using the schemata they know but those schemata develop over time, representing even more complex properties as children get older or more experienced.

According to Morss (1987), children's understanding of perspective develops through three stages. In the first stage, they do not have a specific vantage point to reference the three-dimensionality of spatial environment. A three-dimensional object is attended one face at a time. A child cannot coordinate the different views of an object in this stage. Some time later in the course of the development an object could be viewed and recognized from a single viewpoint. This brings the awareness that an object consists of several faces. A complete coordination of faces (either visible or invisible) is still another achievement that should be made for an accurate representation.

As seen, Mitchelmore's (1976, 1980) findings are consistent with Morss' (1987) theory of children's perspective development. The reason why Mitchelmore (1976, 1980) found four stages instead of three is that the children in fourth stage might have some
kind of training or instruction about the conventions of drawing three-dimensional space figures. In fact, most of the students in the fourth stage were in upper elementary and middle grades. Moreover, the difference between the drawings in third and fourth stages is only parallelism of the lines (see Figure 1).

Cooper's (1990) and Roth and Kosslyn's (1989) findings explain how one processes and stores a three-dimensional object that s/he observes for a short time. Mitchelmore's (1976, 1980) findings and Morss' (1987) theory is more about how one's perspective development changes over a long period of time. They are altogether consistent, although they look from different perspectives. It appears that children pass through three distinct developmental stages while reasoning spatially about a three-dimensional space figure. How this development applies to students' understanding of rectangular solids made of small cubes is elaborated upon in the next section on volume, spatial structuring, and student strategies (Battista & Clements, 1996).

Volume, Spatial Structuring and Student Strategies

Conceptually, there are two kinds of volume: exterior (or displacement) volume and interior volume (Piaget, Inhelder, and Szeminska, 1970). Exterior volume is the space occupied by an object in relation to the surrounding spatial medium. Interior volume is the amount of matter that is contained within a specific boundary, for example, the number of unit bricks in a construction. According to Piaget et al. (1970) children first understand the interior volume but not the measurement of it.

Children should come to understand the logical operation of finding the volume before they are introduced to mathematical (numerical) calculation (Piaget et al., 1970).
Logical operation here is the structuring of the cubes in rectangular buildings as iterable units for enumeration. A schema for logical multiplication can be, first, arrays of squares to be counted one by one and then, second, structured in rows or columns to be iterated. These schemas serve for understanding mathematical operations. Children of four to eight years of age need empirical solutions since they cannot think deductively (Piaget et al., 1970). They cannot do the unit iteration right away. Gradually, they come to substitute mathematical multiplication for logical multiplication. To be meaningful, numerical labeling should correspond to an object. If the student's object is not a three-dimensional unit (i.e., a face of a cube) the volume formula does not make any sense to her or him (Battista & Clements, 1998). As a result meaningful learning does not take place. How students perceive rectangular solids made of small cubes can be inferred from their strategies.

Students use different strategies according to their mental structures available at the time of enumeration while attempting to find the number of cubes in rectangular arrays. This change seems to be developmental and affected by learning the appropriate tasks. Battista and Clements (1996) classified students' strategies into several broad categories. Table 1 depicts major categories of student strategies in dealing with the enumeration of three-dimensional cube arrays. There are several strategies in each category. (See Appendix A for a list of strategies and a detailed description of each strategy).
Table 1

Major Categories of Student Strategies in Dealing with Cube Enumeration Tasks

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
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<tbody>
<tr>
<td>A</td>
<td>Students conceptualize set of cubes as organized sets into layers</td>
</tr>
<tr>
<td>B</td>
<td>Students conceptualize set of cubes as space filling but do not utilize layers for enumeration</td>
</tr>
<tr>
<td>C</td>
<td>Students conceptualize set of cubes in terms of its faces</td>
</tr>
<tr>
<td>D</td>
<td>Students use the formula L x W x H</td>
</tr>
<tr>
<td>E</td>
<td>Other, students do some multiplication but by irrelevant numbers</td>
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</table>

Except for D and E, these strategies are hierarchical in nature and develop from C to A. Strategy E seems to be a misapplication of the formula. Strategy D is a result of the rote memorization of the volume formula without any indication of the intuitive understanding of the measurement of volume. Some steps may not occur or be skipped over very fast by the learner, depending on the instructional experiences. For example, experts use step skipping and abstract planning in developing geometrical proofs (Koedinger & Anderson, 1995). Similarly, as children gain more experience with cube arrays they skip counting schemas and just multiply the dimensions. They count the cubes along the edges just for finding the dimensions. Indeed, people are natively inclined to use more efficient strategies, as they become more experienced in a domain (Siegler, 1987). Spatial structuring is a fundamental notion that provides a theoretical framework for understanding students' strategies in enumerating cube arrays (Battista & Clements, 1996).

Spatial structuring is defined by Battista and Clements (1996) as "the mental act of constructing an organization or form for an object or set of objects" (p. 282). Spatial structuring is a constructive process in which one comes to establish units, relationships between units (such as how they are placed in relation to each other), and a recognition
that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). In their theory of spatial structuring, Battista and Clements (1996) defined four main cognitive milestones for structuring 3-D cube arrays spatially.

Students initially conceive of these 3-D rectangular arrays as an uncoordinated set, or medley, of views. Students at this stage only attend one face of a prism at a time. While they are enumerating cubes, for example, they count the cubes at the outside but not the inside, usually double counting the cubes along the edges. Students who use C strategies (see Appendix A) in their attempts to solve the cube enumeration problems exemplify this conception.

At the second stage, students form composite units and use them in their mental operations. This structuring may or may not be appropriate to solve the problems. For example, if the composite unit is based on the faces of the prism the solution will more likely produce wrong answers. If it is based on layers, rows, or columns as iterable units, the solution will more likely be correct. In other words, composites based on three-dimensional sub-units can be considered proper structuring at this stage of development. Why some students count squares instead of cubes is still not clear. Although they know what a cube is and identify a single cube from its drawing, they may not recognize the cubes in rectangular buildings. It may be that they cannot recognize three-dimensionality from a complex figure.

The third milestone is the coordination. At this stage students are able to coordinate orthogonal views moving beyond the local structuring of the medley of
viewpoint conception of cube arrays. This is the recognition of the fact that different side views are somewhat spatially related to each other and interpreted in relation to each other. This coordination can be achieved on concrete and perspective-like representations but not on separate orthogonal views. This requires different kinds of experience and a higher level of structuring, integration.

*Integration* is the fourth milestone. To integrate views of a 3-D object is to construct a single coherent mental model of the object that possesses these views. A mental model is a perspective-like embodiment that possesses the attended spatial structures of an outside object that are used to simulate interactions with this object. The appearance of this model depends on how the student spatially structured it. That is, it can be individual cubes, an uncoordinated medley of faces, sets of columns, or layers. In other words, a student's mental model of an array is determined by how the student interprets his or her images or views of the array.

The difference between coordination and integration is that coordination implies an awareness of the relation between different views of the object while integration assures that those views have been put together and conceived as a coherent whole in the mind's eye (i.e., a mental model has been formed). Any attempt to think different views in relation to each other can be considered coordination of the views. In order for integration to occur, one should put them together and interpret it as a whole, an isometric-type mental picture. So coordination is a necessary step for proper integration.

Battista and Clements (1996) hypothesized two different processes to explain how the integration occurs operationally. One has to do with recall and the other has to do
with novel object generation. The existing conditions of the learner determine which process will take place. In order for recall to occur one should have similar prior experiences to generate a possible mental model of the whole object by recalling similar objects that s/he had previously perceived or conceived. Then the job of the learner is to fit the pre-existing mental models to the new spatial information to create an appropriate mental model for the new phenomena. The formation of the new image is a kind of transformation of the old images into the new object's shape and dimensions.

If no recalled object seems adequate for a model, a second, alternative process could be activated. In this process, one might start to generate novel objects by performing transformations on images of objects that are available. For instance, one could create an appropriate holistic mental model of a 3-D cube array by visualizing an iterative translation of a single layer through a distance determined by the third dimension of the array. Both recall and object generation processes require the coordination operation. In both situations, one should create the whole from the units either previously encountered or newly generated and check it for appropriateness that requires coordination and integration of different views of the object at hand.

The theory also suggests that the storage and evocation of the objects be done originally, meaning as encountered previously. If for example, one does not have ample experience with different views of an object s/he cannot form an integrated whole from the orthogonal views of a 3-D object. Similarly, if one did not see the layer structure obviously s/he might not be able to recognize this structure too. Therefore, instruction may remedyc this deficit.
Up to now, how students develop three-dimensional understanding, how they structure three-dimensional cube arrays, and what strategies and mental processes they use in this structuring have been emphasized. In other words, the review covered how students could go through this development by themselves. What is missing is that what teachers can do to help students build an understanding of three-dimensional arrays. In what kind of a learning environment and how this structuring is facilitated is sought in the next section.

Stimulating Spatial Structuring

This study has several assumptions about what learning is, how it occurs, and what stimulates it. In this section, I will discuss five assumptions as they apply to students' spatial structuring of 3D cube arrays.

The first assumption is that learning occurs as a result of a change in cognitive structure, and it arises as an independent contribution of the interacting student (Steffe, Thompson, & Glasersfeld, 1998). There is no learning without an individual's attention and intention. In order for learning to occur the learner should actively get involved in solving the problems. Instead of telling the solutions to the students, for example, they should be encouraged to find their own solutions to the problems (Battista & Clements, 1998).

Active involvement can be both physical and mental. According to Gibson (1962), the shape of an object is perhaps its main characteristic. Planarity, curvature, slant, parallellity, span (distance), edges, and corners (vertex) may be conceived as variables of solid geometry. Active touch is an excellent channel of spatial information in
that the arrangement of surfaces is readily picked up. Feeling them best obtains the solid geometry of things. However, active involvement of the learner should not be restricted only to physical manipulation. Sometimes, it can be and should be only mental involvement (Smith, 1998). For example, there is some evidence that at the initial stages of spatial visualization learners need more active control over the spatial objects; later they may benefit more from relatively low active-control conditions such as observations, allowing for a more cognitive source for mental manipulation (Smith, Olkun, & Middleton, 1999).

The second assumption is that instructional tasks are influential components of the learning environment. Representation is a critical factor determining the degree of transfer from one task to another (Anderson, Reder, & Simon, 1996). Many researchers reported evidence that spatial thinking could be improved if appropriate materials were provided (Ben-Chaim, Lappan, & Houang, 1988; Burnet & Lane, 1980; Landau, 1988; Lee & Kingdom, 1996; Lord, 1985). Any effective educational practice should begin with detailed, precise task analysis (Anderson, Reder, & Simon, 1995). The task "finding the number of cubes in rectangular solids" is a spatial, numerical mixed task that demands different types of reasoning, such as spatial, analytic, and numerical. Battista and Clements (1996) reported that while spatial structuring provided input for enumeration, attempts at enumeration engendered spatial structuring. This is an evidence for the mutual interaction between numerical and spatial reasoning. Therefore, activities should demand different types of reasoning in a problem situation to establish a better and richer conceptualization.
The third assumption is that social interaction is an important component of learning (Anderson et al., 1995, 1996; Cobb, 1994; Cobb, Yackel, & Wood, 1992). Through interaction with peers and adults students appropriate their knowledge and come to a common understanding of the problem situation, which is necessary for communicating mathematically. The experience children gain through interactions with adults such as a facilitative teacher greatly influences their construction of mathematical knowledge (Cobb & Steffe, 1983). In a study by Empson (1995), for example, the teacher was able to help children begin to formalize their intuitive knowledge by structuring the instructional interactions in such a way that children were provided with resources to assist their thinking, and by involving them in conversations about their solutions. Asking questions that constrain students to direct their attention to certain aspects of the problem might be a good strategy for stimulating learning (Anderson et al., 1995). Students may change their understanding sometimes by merely imitating others for awhile and/or negotiating about the meaning.

The fourth assumption is about the need for sequencing the instruction. According to Shuell (1990), there are phases in learning. These phases can be identified. The transitions through phases can be facilitated by appropriating the sequence of stimuli. Inappropriate materials do not only prevent learning but also kill motivation and affect subsequent learning (Anderson et al., 1995). The task of the educator is, therefore, to find out where the students are at and to design a series of experiences for students that will enable them to learn effectively and to motivate them to engage in the corresponding activities (Anderson et al., 1995). Previous research shows that while finding the number
of unit cubes students are relatively more successful on small size buildings and problems with concrete representations (Battista & Clements, 1996; Ben-Chaim et al., 1985). Activities can be sequenced according to this finding.

The fifth assumption is that students' prior knowledge (including informal and intuitive knowledge) plays an important role in their further understanding (Hiebert & Carpenter, 1992). Understanding involves establishing relationships between segments of knowledge (Fennema, Carpenter, & Peterson, undated). Therefore, any components of the learning environment should be carefully adjusted to the purpose of evoking students' prior knowledge. Instructional interactions enable children to gain access to resources and tools that both reorganize and transform their thinking, resulting in more powerful cognitive structures. While solving problems using their prior knowledge, children discover new ways of looking at things, thus leading the development of that knowledge into more sophisticated ones. For example, layer structure is not inherent in 3D arrays of cubes (Cobb et al., 1992). It is an abstraction. Children gradually construct the layer structure while they are solving problems and acting on similar objects involving layers. Activities can be arranged to stimulate this transition.

One way to initiate students' intuitive knowledge would be to provide a semantically rich context such as equal sharing. Empson (1995), among others, argued that students had intuitive knowledge about equal sharing. In her study, even first graders were able to solve problems that involved fraction concepts by calling on intuitive knowledge of equal sharing. Similarly, students might be able to initiate their intuitive knowledge in a similar context. For example, through equal sharing of spatial
constructions they may develop the understanding that 3D arrays of cubes consist of layers that have equal amounts of cubes in them by attending specifically to the layer structure of buildings. While they are engaging in constructing and deconstructing cube buildings with equal amounts such as layers and columns, they may abstract the equality of layers and transfer this knowledge into tasks such as cube enumeration. This development may lead them to form spatial and numerical composite units and unit iteration schemas. Then, through conversations with peers and teacher they may appropriate and formalize their knowledge.

To summarize, learning is a long constructive and cumulative process that occurs as a result of changes in cognitive structures that are caused by mental and physical actions of the learner. The learning environment, including the tasks and materials, also has an effect on the rate of this change. Social structure of the environment in which education takes place is of utmost importance from a cognitive, and especially from a motivational, standpoint (Anderson et al., 1995).

As seen, there is a substantial amount of information about what the necessary conditions of learning should be: active learner, social interaction such as peer interaction and facilitative teacher, appropriate tasks, and context for initiating intuitive knowledge. However, there is no report on how an instruction based on these assumptions might result in cognitive change in students' spatial structuring.

Summary

Finding the number of cubes in rectangular solids provides cognitive framework for understanding the measurement of volume. It is a three-dimensional, spatial,
numerical mixed task that requires visualization. Visualization inherently involves reasoning of different kinds such as spatial and analytic. There is a synergistic effect between numerical reasoning and learning geometry. Consistent with the Piagetian theory of cognitive development, at around the age of seven to eight years old, children have a global understanding of three-dimensionality. Nevertheless, they struggle with the task of counting the number of cubes in three-dimensional rectangular arrays.

Initially, children only focus on the cube arrays in terms of its faces. While enumerating cubes they count the cubes at the outer faces of the prism, sometimes double counting the cubes along the edges and triple counting the cubes in the corners. At the second stage, children recognize the cubes as whole entities but not as systematic configurations. Therefore, they usually try to count the cubes one-by-one and lose track of their counting, and again some double counting occurs. Then, they start to form some organization of the cubes but these structural entities are not systematic yet. At the third stage, they can form some kind of iterable units such as layers, rows, and columns and use them in their enumeration.

There is much information on how to stimulate students' learning and spatial thinking in general and spatial structuring in particular. Active learner, social interaction such as peer interaction and facilitative teacher, appropriate tasks, and context for initiating intuitive knowledge are the components that seem most relevant to student learning in general and spatial structuring in particular.

As we have seen, previous research has identified how an individual structures the arrays of cubes and what are the necessary mental constructions for a proper structuring.
However, how an instruction based on these premises will result in cognitive change considering the students’ spatial structuring has not yet been studied.

Research Questions

_Problem statement:_ How do the activities based on equal sharing of 3D spatial constructions affect (4th grade) students' understanding of rectangular solids made of small cubes? Following subquestions are posed to answer the main question:

1. What strategies do students use to find the number of unit cubes contained in rectangular solids before the instructional unit?
   a. How do they view the buildings?
   b. Do they use any layering strategies?
   c. Do they perform any unit iteration schemas?
   d. Is there any change in strategies with differing number of cubes?
   e. Do they use the same strategy in concrete and pictorial situations?

2. How do students interact with the activities based on equal sharing of 3D spatial constructions during the course of the instruction?
   a. How do they approach equal sharing problem situations?
   b. How do they partition the buildings?
   c. Do they form any composite unites?
   d. What kind of information do they use first and more spatial or numerical?
   e. In what direction does this information use change?
   f. Do they develop any unit iteration schemas?
   g. Do they invent any layering strategies?
3. What strategies do students use to find the number of unit cubes contained in rectangular solids after the instructional unit?

   a. How do they view the buildings?
   
   b. Do they use any layering strategies?
   
   c. Do they perform any unit iteration scheme?
   
   d. Is there any change in strategies with differing number of cubes?
   
   e. Do they use the same strategy in concrete and pictorial situations?

4. Why do the students use the strategies they use and how do the strategies change?
CHAPTER 3
METHODS

This study utilized a quasi-experimental design but not in a classical sense (Campbell & Stanley, 1966). The intent of the study was not to obtain a statistical end result in a strictly controlled environment in which variables were manipulated and their effects upon other variables were observed. Rather, it aimed at identifying students' understanding from one point in time to a second point in time, during which time the children were stimulated by a series of activities. The transition from one stage to the other was observed and documented as detailed as possible for the children's underlying mental structures and their engagement in solving problems as a result of the provocative stimuli. In other words, what the students did was of concern, but of greater concern was how they did it (Cobb & Steffe, 1983) and why they did it, or what provoked it in the environment.

In many respects, the method of this study is similar to the teaching experiment methodology (Cobb & Steffe, 1983; Steffe et al., 1998). For example, in the instructional period different hypotheses were tested about how students could make sense out of sharing problems. Although there is not a standardized form for teaching experiments, the experimental method employed in this study does not fit exactly into the teaching experiment methodology. For example, this study looks the effect of environmental stimuli that is not a concern in a teaching experiment.

This study sought the answer for the question, "Does the successful completion of equal sharing problems involving cube buildings lead to a better strategy use in cube
enumeration tasks?" If "yes," "how did this process occur?" Therefore, the process was as important as the product in this experimental study.

The experimental design included three phases (pre interviews, instructional intervention, and post interviews) in which an in-depth analysis of students' learning was sought with a few participants rather than a shallow end result with a lot of students. The task of the first phase was to identify the students in terms of their level of functioning with the cube enumeration tasks. The purpose of the second phase was to try to affect students' understanding by equal sharing word problems provided with concrete and pictorial materials as well as testing the hypothesis generated after the first phase. Another task in the second phase was to observe and document students' interactions with the tasks and others to account for the provocative stimuli in the learning environment. Then, the task of the last phase was to reassess the students' level of functioning to see if any changes occurred in students' approach to the same cube enumeration tasks used in the pre-interviews after the completion of equal sharing problems.

Before going into the field, the researcher prepared the interview questions and the instructional activities based on the hypothesis generated during the review of the literature. Then, a pilot study was carried out to see if the participants, questions, activities, and the setting were all suitable for the study. The intended format for the methods, the revisions during the pilot study, and the actualized format are described in detail in the subsequent sections.
Participants

All participants were solicited from a charter school in the greater Phoenix metropolitan area. The school had no special interest in terms of curriculum. It was not different from a regular public school. Except for convenience, there were no particular reasons to choose the school for the study. However, it should be mentioned that the majority of the students in the school were minority and from a low socioeconomic status. There was only one class from each grade level in the school. Student participation was on a voluntary basis.

There were two criteria for selecting the participants. One had to do with grade level. Measurement of volume is introduced around the fifth grade. Since this study assumed that an understanding of the logic should come before the introduction of the volume formula, participants were chosen from the third and/or fourth grades. Another criterion for selecting the students was that they would be average or slightly below average students in mathematics. This criterion was set forward just to make sure they were not special students.

For the pilot study, three fourth graders (one above average, one average, and one below average student in mathematics), and two third graders (one average and one below average) were selected from two mathematics classes. There were about 17 third graders and 13 fourth graders in the school. Since the students at that time did not have any standardized mathematics tests, their classroom teachers, based on their perceptions, were asked to assign the students. Except one third-grader, the participants in the pilot study were girls. During the pilot study, the researcher felt that the activities were not
appropriate for third graders since they could not differentiate the cube and cube faces even in a very simple situation (i.e., one-layer pictorial building with 4 cubes). It was obvious they needed more experience with concrete cubes and their drawings in simpler situations. Therefore, participants for the final study were chosen from the fourth graders, again based on their classroom teacher’s perceptions.

The teacher was asked to choose the students who were average or slightly below average in mathematics. A total of four 4th grade students, two girls and two boys, all from low-income families, participated in the final study. The following specific characteristics of the participants were obtained from their teachers during the informal talks with him and interviews with the participant students.

DA was an African-American boy. He was a fourth grader and was described by his teacher as a below average student in mathematics. DA said he had never seen such cubes before; however, he had seen larger ones and used them as counters. He had also played a little bit with Legos.

CH was a Caucasian boy. He was a fourth grader and was described by his teacher as a below average student in mathematics but very high in verbal ability. He said he had seen similar cubes before, and except for Legos he did not have similar toys.

SA was an African-American girl. She was a fourth grader and was described by her teacher as a below average student in mathematics. She said she had never seen such cubes before. She had played minimally with Legos as a small child.

ST was a Hispanic girl but she did not have problems with her English. She was a fourth grader. Her teacher described her as an average student in mathematics. ST said
she had not seen these type of cubes before. She had used similar ones, not for building, but as loose cubes for counting and addition in the first grade. She had also played minimally with Legos.

**Materials**

Four types of materials were used for the interview and instructional purposes: colored pens, wooden cubes (a centimeter along each edge), rectangular buildings made by gluing individual wooden cubes together (the same size as the loose cubes), and drawings of the concrete buildings. Colored pens were used by the participants to show the different shares by shading parts of the buildings in the drawings. There were about a hundred loose wooden centimeter cubes. They were used for building purposes. Seven rectangular buildings composed of a varying number of cubes, that is, 8, 9, 12, 16, 24, 36, and 48 cubes (see Appendix C) were used as a supplement to the pictorial representations. That is, students who failed to do the task with pictures returned to the concrete buildings in order to complete the task. Drawings of these buildings were also available during the study and used according to the aims of the study such as for shading different shares and comparing with the concrete buildings.

There were other materials used for recording purposes such as a video camera with sound and picture-capable, videotapes, tripod, and paper and pencil. Three 120-minute videotapes were recorded during the study. Additionally, students were given candy bars and teddy bears as incentives after each session, both in clinical interviews and instructional sessions.
Clinical Interviews

Clement (1998) argued that through clinical interviews it was possible to collect and analyze the kind of data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in a subject's thinking. In this study, clinical interviews were intended to determine each student's ways of thinking about the cube configurations as presented both pictorially such as in textbooks and national examinations (see Appendix B) and concretely (concrete representations of the same buildings depicted in Appendix B). There were two phases of interviews, pre- and post-clinical interviews. The same protocol (see Appendix B) was used in both pre- and post-interviews.

Participants were individually interviewed after the lunch break during the mathematics class in a 5x6 meter room separate from the classroom. There were two doors in the room that accessed the restroom and the outside. In the room were four chairs and a round table in the room. The interviewer and the interviewee sat around the table face to face. The video camera was placed on a tripod, 2.5 meters away from the table, and the cameraman silently stood behind the camera. Camera light was not used during the interviews, because it might have diverted the participants' attention. Before each interview, each participant was allowed to say and do something in front of the camera in order to reduce the novelty and make them familiar with the setting. After the familiarization, they were given the questions. Not long after the interviews commenced, the children were so on task that they did not even look at the camera.
Each clinical interview was videotaped to account for all the actions and verbalizations of the participants while they were attempting to enumerate the cubes in rectangular buildings both in pictorial and concrete situations. Retrospective reports were additionally used if the strategy could not be determined through observation by the interviewer. Ericcson and Simon (1993) suggested that for brief solution processes the recalled sequence of thoughts could be reported retrospectively with high accuracy, with only the effect that memory for the reported information was strengthened.

Participants were asked to find the number of cubes in rectangular buildings using first graphical and then concrete representations of rectangular buildings made of small cubes. They were asked to solve the problems with concrete cubes after they finished all the questions with pictorial representations. In volume related tasks, although the differences between the modes were not statistically significant, the third and fifth graders' overall scores in problems with concrete representations were slightly better than with the graphical representations (Battista & Clements, 1996; 1998; Phillips, 1972). Therefore, in the clinical interviews, both modes of presentations were used to detect if there were any variations in the strategies used by the participants.

Through clinical interviews, the strategies used by the students while solving the cube enumeration tasks were obtained in a variety of situations in order to elicit multiple levels of sophistication in solution strategies. Situations included the questions with one-layer, small, medium, and large buildings in both concrete and pictorial representations (see Appendix B). The pre-interview data were thoroughly examined and partly analyzed before the intervention in order to provide further insight into the instructional
intervention phase. Based on these results, some hypotheses were generated and then tested in the instructional intervention period.

The hypotheses were concerned about the complexity of tasks and the children's level of understanding. One hypothesis was, for novices, the complexity of the cube buildings increased with the size of the building, and pictorial representations were more complex than their concrete counterparts. Participants of this study did not have any difficulty acting on a one-layer building. The second hypothesis was concerned about the three levels of understanding of rectangular buildings made of small cubes (Battista & Clements, 1996).

Post-clinical interviews were conducted with each participant six days after the last session of his or her intervention. The reason for these interviews was to probe if any improvement in strategy use was made due to the instructional intervention. The procedural format for the post-clinical interviews was exactly the same as the pre-clinical interviews.

Development of Instructional Activities

Instructional activities were intended to further students' understanding of rectangular arrays of cubes towards layering strategies. To be effective, activities were based on what was already known about the children's learning and what the students were already capable of doing. Three sources of information were used in the development of the instructional activities: previous research, the pilot study, and clinical interviews.
After having described the development of student understanding of 3D arrays, Battista and Clements (1996, 1998) proposed some activities and instructional guidelines that might help students develop more powerful ways of thinking about these problems. Two of their findings, among others, seemed very crucial. First, they thoroughly documented that the most viable student strategies for the enumeration of cubes in buildings were the layering strategies, and that these took a long time to be established. Second, they also suggested that students' attempts at enumeration engendered their spatial structuring of cube arrays while their spatial structuring led them to develop better enumeration strategies. This finding lent support for the existence of a mutual interaction or a synergistic effect between numerical and spatial reasoning (Clements et al., 1997; Lubinski & Humpfreys, 1990). Therefore, it seemed reasonable to assume that some specific activities that could pinpoint the layer structure of cube buildings in a problem environment which emphasized the numerical and spatial aspects of the buildings could help students develop a better understanding.

Previous research also showed that it was possible to initiate students' intuitive knowledge in similar contexts. Equal sharing is one of the contexts students have intuitive knowledge about (Empson, 1995). Keeping these principles in mind, the activities for the instructional intervention were produced and field-tested in a pilot study.

Initially, intervention consisted of equal investment and equal sharing word problems that invited a pair of students to make equal investments to make buildings with different numbers of cubes or share the buildings equally between a certain number of people. Students have intuitive knowledge of equal sharing (Empson, 1995). Therefore,
they might be able to initiate this intuitive knowledge while solving the problems involving the equal sharing of cube buildings. This experience may lead them to attend the spatial structural elements of cube buildings. Equal investment might demand the use of this intuitive knowledge in the other direction. Shares would be different layers, rows, and columns that are the structural elements of buildings.

The main idea behind these activities was threefold: First, students systematically construct rectangular buildings made of small cubes using their structural elements such as cubes and some units of cubes (rows, columns, and layers). Second, they deconstruct them back into their constituent parts such as layers, rows, columns, and individual cubes in equal shares. Third, while they are trying to solve the problems they will have the chance of both numerical and spatial reasoning about investing and sharing situations. There is evidence that mathematical thought can be stimulated by providing visual clues to the observer (Nelsen, 1993) or the other way around (Battista & Clements, 1996). That is, a numerical demand may constrain the participants to attend to certain visual clues or visual clues may stimulate a more sophisticated numerical reasoning.

In the activities provided, there was a numerical target for stimulating a spatial solution or a spatial target for a numerical solution. For example, while sharing eight cubes contained in a rectangular solid a student can either say "8 divided by 2 is 4" or s/he can see the two slices of four cubes and may claim their equality by placing them side-by-side. In either case, students can be encouraged to see the other solution too.
The purpose is to provide the students with both numerical and spatial information and have them feel the need for a solution that is satisfactory.

Activities were produced and piloted prior to the final study. During the pilot study, it was felt that participants were not very comfortable with the word "investment" so it was replaced with the phrase "putting or giving equal shares." For example the sentence read, "Let's make this building with four people by each of us giving equal shares." Students in the pilot study did not hesitate using the idea of equal sharing. It seemed very natural to them.

Pre-clinical interviews were another source that helped construct the activities. For example, gradually increasing the size of the buildings and using different modes of presentations simultaneously were inspired from the hypotheses produced after a thorough examination of the data from the pre-clinical interviews. Participants were not consistent in their use of strategies for different size buildings and different modes of presentations. That is, they used different strategies (which reflected their conceptualizations) in different situations. This information gave way to the hypothesis about the complexity of the problems. The fact that they did not have any difficulty with a one-layer building, either concrete or pictorial, triggered the idea that pinpointing repeatable units could be very helpful for establishing patterns in the buildings based on those elements such as layers and columns. Pre-clinical interviews also showed that there could be major differences in terms of difficulty between the pictorial and concrete buildings. All these aspects of information were used in the development of the instructional activities.
The activities used in the final study are depicted in Appendix C. The tasks included in each activity were as follows: (a) constructing a building with loose, wooden cubes by looking at its picture and comparing them for their equality, (b) partitioning the building made with loose cubes into equal shares, (c) partitioning the concrete building into equal shares and showing one share, (d) partitioning the pictorial building into equal shares and showing one share, (e) coloring shares in different colors on the drawing, (f) finding other ways of partitioning the same building in both concrete and pictorial representations.

Instructional Intervention

The purpose of the instructional intervention was to promote students' understanding of rectangular buildings made of small cubes by providing them appropriate materials and environment in which they could interact and further their understanding. The main elements of the instructional environment were the activities, materials, peer-interaction, and instructional guidance by the teacher/researcher.

Students were videotaped during the intervention to account for their full interaction with the learning environment. The same room was used for the instructional interventions. This time, however, students interacted in a group. In the pilot study, two groups were made: one from three 4th graders and another from two 3rd graders. Initial interactions during the pilot study showed that it was hard and confusing to keep track of the participants' conversations while they were in a group of three. The third student did not seem to be involved. Pairs seemed more appropriate, and were used in the final study. Two boys (CH & DA) and two girls (SA & ST) were assigned in two different
groups. Other than gender, there were no particular reasons for assigning them as boys and girls because their teachers described them all as below average students, only ST being close to average.

Each group was instructed at different times. The teacher/researcher and students sat around a table in the order of teacher/researcher, student 1, and student 2. On a tripod, 2.5 meters away from the table, was a video camera pointing to the top center of the table. Again, the camera light was not used. The teacher/researcher introduced the first activity by giving each student a sheet of paper that had the question and the drawing of a 2x2x2 cube building on it (see Appendix C).

Based on the hypothesis (about the complexity of tasks) generated after the pre-interview results, a six-step procedure was used for each activity. The students were provided first with the question and the drawings of buildings to be shared. Then, they were told to read the question and show the solution on the picture. For example, activity one read as follows:

You have this building [showing the drawing of 2x2x2 building]. You want to share it equally between 2 people. How would you do that?

The six steps to Activity 1 are illustrated as follows:

Step 1: Can you show me one share on the picture?

That way, students had a chance to mentally imagine the solution and predict an answer in their mind. Second, they were asked to shade the shares in different colors on the drawing.

Step 2: Can you shade each share in different color on the picture?
If no attempt could be made or the student did it wrong then they were additionally provided with the concrete building and asked:

*Step 3: Can you show me one share on the building?*

If there was still no attempt, then they were given loose cubes and encouraged to construct the building by looking at the picture while trying to partition it.

*Step 4: Can you make the building with cubes? Look at the picture and make exactly the same building.*

If there was still no attempt then the teacher/researcher, as a fifth step, made one share out of cubes, asked them to do the other share, and then had them put the shares together and take them apart. As seen, the tasks were introduced from more complex to less in order to find out what the students could and could not do. After that, the first four steps were passed through in reverse order to make sure that the students accurately completed the task. For example, students were asked to construct the building with loose cubes. They were directed to partition the building and then to show one share on the concrete building and one share on the drawing respectively. Then, they were required to color the shares in different colors in order to have them make their works more explicit on drawings.

After they finished the partitioning in one way, they were asked to do the task in a different way. This task enabled them to look at the building in different ways, such as front-to-back layering, bottom-up layering, or left-to-right layering etc.

*Step 6: Can you do this another way on the concrete building and show me by taking the equal shares apart?*
All these procedures together provided a flexible environment in which the students could solve the problems in one way or another (mode of presentation), transfer his or her knowledge into other modes of presentations, and try different ways of doing the task. For each activity, attempts were made to follow the same procedure. However, not all the steps were needed in all of the activities. Unexpected student interactions during the course of the instruction might have been a reason to skip one or more of the steps. If, for example, one of the partners was able to do the task in the first step and trying to explain it to the other partner they were not interrupted. The majority of the learning environment was not strictly controlled (on purpose), so the students were left to do the task. However, they were guided, if the teacher/researcher felt it necessary.

Each successive activity introduced a larger building that made the task a little bit more complex than the previous one. That is, the size of the building was gradually increased to provide the students with problems that were one step further than their current capability. This procedure was derived from the hypothesis that, for novices, the size of the building affects the complexity of the building. Initial evidences for this hypothesis were gathered from the results of the pre-clinical interviews. For example, they used different strategies as the size of the building increased.

Timing

After having been approved by the Institutional Review Board, the consent forms were given to the students on February 1, 1999. The data generation started by informal talks with the teacher and continued by individual pre-interviews with two students (SA & ST) who brought in their parental consent forms on February 3. On February 4 and 5,
they received a 101-minute instruction as the first pair of the two groups in two sessions of approximately equal length sessions: 52 minutes, first session; and 49 minutes, second session. On February 5, students in the second pair (CH & DA) were pre-interviewed individually after they brought in their parental consent forms. Their instruction lasted a total of 65 minutes, 42 minutes for the first session and 23 minutes for the second session. Post-interviews were given to each participant individually six days after the last day of their intervention.

A summary about the timing is given in Table 2. As seen in the table, the duration of the pre- and post-interviews varied from 3 to 11 minutes depending on the student interviewed and the pre-post condition. All of the participants spent less time in the post-interviews. Boys (CH & DA) spent less time than did girls (SA & ST) in the intervention period although the boys solved one more problem than did girls.

**Table 2**

**Timing of Interviews and Intervention (in Minutes)**

<table>
<thead>
<tr>
<th>Participant</th>
<th>Pre Int.</th>
<th>Intervention</th>
<th>Post Int.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ST</td>
<td>11</td>
<td>53+49</td>
<td>4</td>
</tr>
<tr>
<td>SA</td>
<td>9</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>CH</td>
<td>6</td>
<td>42+23</td>
<td>3</td>
</tr>
<tr>
<td>DA</td>
<td>6</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>TOTAL</td>
<td>32</td>
<td>166</td>
<td>14</td>
</tr>
</tbody>
</table>

In both pre- and post-interviews, questions were read to participants by the interviewer/researcher. Then, participants were left to find the number of cubes in the building. They spent as much time as they wished. When they found an answer, the interviewer/researcher asked them to justify their answer "Can you tell me how you
got that?" Sometimes, however, if their answer were so obvious, the interviewer/researcher did not ask them to justify their answer. This happened more often in the post interviews. So, the total time elapsed in post-interviews decreased for this reason. However, the main reason that made the total time shorter in post-interviews was that the participants developed more efficient strategies as explained in the "Results" section.

The time difference between boys and girls during the intervention occurred because the boys seemed to be faster in understanding the problem and devising a solution to it than did the girls. Although their baseline data showed that both boys and girls were approximately at the same level of functioning and their post-interview data showed they were at the same level of understanding after the intervention, the boys spent less time than did the girls to reach approximately the same level of improvement.

Analysis

A qualitative interpretive (Ericksson, 1986) framework was used in the analysis of the data. Data analysis began during data generation when some coding and memo writing were done as part of the analysis. The majority of the data was generated through clinical interviews and observation of the instructional periods. To enable the repeated retrospective analysis, three 120-minute videotapes were recorded. Field notes were also made during the interviews and instructional periods.

Data analysis occurred daily after the clinical interviews and observation of the instructional periods. The viewing of videotapes and reviewing of field notes occurred
daily immediately after the sessions to enable the production of working hypotheses. Then later the videotapes were reviewed several times to make sense of the overall picture. Students' actions and verbalizations were fully transcribed from the clinical interviews. Field notes were inserted into the transcription. A total of 35 pages of 10 point single-spaced, verbal data plus 76 pictures of the students' work in a step-by-step format were generated through these transcriptions (see Appendix D).

To reveal the patterns in the data, the transcripts and the student works on the drawings were read and examined several times. During the data generation, the transcription phase, and the multiple reading phase, some coding and memo writing were also done as part of the analysis. For example, the data were coded according to the points at which discernable leaps were made by the participants. Relevant interpretations were made regarding the underlying mental operation that was thought to cause the action to happen and the external stimuli that were thought to provoke the action to happen.

Battista and Clements (1996) documented with detailed descriptions the strategies of third through fifth grade students while they were enumerating arrays of cubes (see Appendix A). In this study, the documented strategies were also used to determine the students' level of reasoning about cube enumeration tasks during the two clinical interview phases. First, the students' actions and verbalizations were fully transcribed for each student and for each problem in the interview protocol (see Appendix D). Then, the students' strategies for each question were labeled according to the categories produced by Battista and Clements (1996), and then tabulated. All the strategies used by the
students fell into these categories. Some specific or extreme cases are discussed in the "Results" section as well.

A more open-ended format was used to analyze the data generated during the intervention period to account for the newly emerging constructions made by the participant students. After the data collection, some empirical assertions were stated based on the emerging patterns in the data. Then, each assertion was warranted by bringing empirical evidences from the data generated during the interviews and instructional periods.

In the following section, how answers to each particular research question were sought is explained in detail.

Research Question 1: What strategies do students use to find the number of unit cubes contained in rectangular solids before the instructional unit?
   a. How do they view the buildings?
   b. Do they use any layering strategies?
   c. Do they perform any unit iteration schemas?
   d. Is there any change in strategies with differing number of cubes?
   e. Do they use the same strategy in concrete and pictorial situations?

The purpose of this question was to establish baseline data for further interventions by determining the students' level of functioning on the cube enumeration tasks in different situations based on the strategies they used. There were eight different situations determined by the size of the buildings and the mode of presentation. Sizes were labeled as one-layer, small, medium, and large building (see Appendix B). Two modes of presentations were the pictorial and the concrete situations.
Research Question 2: How do students interact with the activities based on equal sharing of 3D spatial constructions during the course of the instruction?
   a. How do they approach the equal sharing problem situations?
   b. How do they partition the buildings?
   c. Do they form any composite units?
   d. What kind of information do they use first and more spatial or numerical?
   e. In what direction does this information use change?
   f. Do they develop any unit iteration schemas?
   g. Do they invent any layering strategies?

This question was intended to uncover the meanings participants held about the problems and their solutions. The answer to this question was sought in their actions, verbalizations and works documented during the intervention. For example, it was discovered that participants exhibited the same three levels of conceptualizations while acting on the cube buildings. This finding enabled the use of spatial structuring theory (Battista & Clements, 1996) as a supplementary theoretical framework in analyzing the intervention data.

Research Question 3: What strategies do students use to find the number of unit cubes contained in rectangular solids after the instructional unit?
   a. How do they view the buildings?
   b. Do they use any layering strategies?
   c. Do they perform any unit iteration scheme?
   d. Is there any change in strategies with differing number of cubes?
   e. Do they use the same strategy in concrete and pictorial situations?

The aim of this question was to account for any change overall in a student's approach to the same cube enumeration tasks due to the instructional intervention. Results were compared to the ones obtained during the pre-interviews.
Research Question 4: Why do the students use the strategies they use and how do they change?

The answer to this question was sought in students' strategies that they used in different situations. It was a purely interpretive endeavor to attach meaning to their strategies about why they used the strategies they used and how they changed.

Answering this question also enabled the researcher to find possible internal effects that were thought to cause cognitive change and the external effects in the learning environment that were thought to provoke students' understanding.
CHAPTER 4

RESULTS

After multiple readings and examinations of the data generated during the study the following empirical assertions are stated based on the emerging patterns in the data:

1. Before the instruction, participants used different strategies depending on the complexity of the buildings. Complexity increased with the size of the building. Pictorial representations were also more complex than their concrete counterparts.

2. In the pre interview, re-attempts at enumeration usually resulted in an improvement and/or change in strategies. Sometimes, however, it was detrimental.

3. Instruction caused a consistent and long lasting change in student strategies regardless of the building size and concrete versus pictorial situations. However, students' level of reasoning in numerical domain may have restricted any further development.

4. Changes in student strategies increased student confidence and solution accuracy and decreased the total time to solve the interview questions.

5. Counting the visible cube faces strategy (C1) was unique to pictorial situations and seemed to be related to the perception of three-dimensionality of rectangular arrays of cubes from their drawings.

6. Initially both numerical and visual clues were used in solving problems but the instruction increased the use of visual clues.
7. During the instruction participants progressed through three different conceptualizations of rectangular solids made of small cubes.

8. During the instruction, three kinds of interaction contributed to the invention of new, more viable strategies.

These assertions are warranted in the subsequent sections by bringing empirical evidences from the data generated during the interviews and instructional periods.

Assertion 1

_Before the instruction, participants used different strategies depending on the complexity of the buildings. Complexity increased with the size of the building. Pictorial representations were also more complex than their concrete counterparts._

All of the students interviewed enumerated correctly the cubes in Question 1 in both graphical (a 2x2x1 building) and concrete (a 3x3x1 building) situations. This means that they comprehended both concrete and pictorial representations of one-layer buildings. Since Question 1 (concrete and graphical) had all the cubes visible, there was no ambiguity created on the part of participants. That was why it was eliminated from further analysis; however this knowledge and the level of functioning seemed to be prerequisite for further understanding of rectangular solids made of small cubes.

As shown in Table 3, all the strategies used by the participants for enumerating rectangular buildings made of small cubes fell into categories made by Battista and Clements (1996). (See Appendix A for a detailed descriptions of student strategies). Not one student used Category D or Category E strategies where the student used the formula
rotely or misapplied it. This was because the participants had not yet been introduced to the volume formula.

The number of unit cubes in the building or the size of the building in three dimensions affected the students' strategy choices. All but one student used Category A strategies for Question 2 (a 2x2x2 building) in both concrete and pictorial situations in the pre-interview (see Table 3). For the fourth question (a 4x3x3 building), however, two students used Category C strategies, one A and one B strategy in concrete situations. This difference was even greater in the graphical situation. All but one student used C strategies for Question 4 (pictorial). One student used a B strategy for the same question. It seemed that when the size of the building increased the task became more complex or overwhelming for the students. If they did not have a consistent strategy, then they used less viable strategies, such as C strategies in order to find a solution to the problem at hand.

Table 3

<table>
<thead>
<tr>
<th>Student</th>
<th>Test</th>
<th>Concrete</th>
<th></th>
<th>Graphical</th>
<th></th>
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<th></th>
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<tbody>
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<td>4</td>
<td>2</td>
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<td>2</td>
</tr>
<tr>
<td>SA</td>
<td>Pre</td>
<td>A2</td>
<td>A2</td>
<td>C3</td>
<td>A2</td>
<td>B1</td>
<td>C3</td>
<td></td>
</tr>
<tr>
<td>ST</td>
<td>Pre</td>
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<td>C3</td>
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<td>C1</td>
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<tr>
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<td>A3</td>
<td>A3</td>
<td>C1</td>
<td>C1</td>
<td>C1</td>
<td></td>
</tr>
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<td>CH</td>
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<td>A2</td>
<td>A2</td>
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</table>

Note. For the explanation of each strategy, see Appendix A.
Participants also used different strategies for concrete and pictorial situations although the same size buildings were used in both of the situations. Students chose Category A strategies more often (8 times in 12) in concrete situations than they did (4 times in 12) in graphical situations. Alternately, they used Category C strategies more often (6 times in 12) in graphical situations than they did (2 times in 12) in concrete situations. This difference was even greater for some students, for example, DA and ST, than it was for the other two students.

DA did not make a single mistake nor show any hesitation in determining the number of cubes in one-layer buildings, both concrete and pictorial. He was able to do correctly all the problems presented concretely using Category A strategies, however, he used C1 strategy and he did not obtain any correct from the pictorially presented problems. It seemed that he almost completely lacked understanding of three-dimensionality in graphical representations. While finding the number of cubes in pictorial buildings, he consistently counted only the visible cube faces, a typical C1 strategy. (There is more about C1 strategy in Assertion 5).

The effect of the size of the building and the effect of pictorial condition were also evident during the intervention. Each time the size of the building increased some students, such as DA and ST, seemed overwhelmed, and thus the researcher/teacher had to return to a simpler situation. In addition, students solved the problems with concrete materials more easily than the problems with graphical representations. Therefore, most of the time while working on pictorial representations, the researcher/teacher had to go back to concrete materials, solve the problems with the students, and then compare the
construction with the pictorial representation. At the very beginning of the intervention it
took the students some time to work with pictorial representations at the same level of
ease with which they worked with concrete materials. Towards the end of the
intervention, however, there were no such difficulties.

This difficulty can be explained by the fact that the pictorial situations were
perceptually more difficult than the concrete situations (Orton & Frobisher, 1996).
Additionally, students, at least participating in this study, were relatively more familiar
with the concrete materials than they were with pictorial representations of cube
buildings. Similar results were reported in the literature (e.g., Battista & Clements, 1996;
Ben-Chaim et al., 1985). However, even the experience they had during the intervention
was significant in students' acquisition of the skill to accurately interpret two-dimensional
graphical representations of certain three-dimensional objects.

These findings have important implications for instruction. First, it was possible
to determine the students' level of reasoning in similar tasks. It was assumed that the
children's observed actions were performed for a reason that made sense to them
(Wheatley & Cobb, 1990). If students' strategies reflected their available conceptual
structures, then it was possible to sequence the instructional tasks according to the needs
of individual students.

Assertion 2

*In the pre-interview, re-attempts at enumeration usually resulted in an
improvement and/or change in strategies. Sometimes, however it was detrimental.*
In the pre-interview, each time the interviewer asked, "Can you tell me how did you found that?" students, who used some B and C strategies sometimes found different results because they could not keep track of their counting. So they changed their answers several times until they felt that the answer was correct as evidenced in Episodes 1 and 2.

Episode 1: CH is solving Question 3 (graphical). His strategy is categorized B2 (a subcategory of B).

CH: [Counted the columns of cubes starting from the far left. He tapped twice to each cube in the row to account for the cubes at the bottom] 12.

Int: Can you show me how you did that?

CH: [He did the same thing but more carefully and this time he got it correct] 16.

Int: How do you think that you missed some in the first trial?

CH: I counted the back but in the front I didn't count these [showing the front bottom row].

Episode 2: ST is solving Question 3 (concrete). She started with C3 strategy but switched to B3 strategy.
ST: [Counting the cubes all around the building] 17
Int: Can you show me how you got that?

ST: 1, 2 . . . 8 [counting the cubes in the top layer] and 1, 2, 3, 4 here [counting the cubes in the bottom right row] and 1, 2, 3, 4 here [for the left row; after turning the building around], 8 [she forgot the first group of 8 cubes].

Int: So, you have two different answers, 8 and 17. Which one do you think is correct?

ST: 8.

Int: Can you show me the 8 cubes again?

ST: 1, 2 . . . 8 [for the top layer] 9, 10, 11, 12 [she started counting on, while counting the right bottom row] 13, 14, 15, 16, 17. Wait, no, 16 [she first double counted the cube at the far end of the building but realized that it must be 16].

She became more systematic in her counting. She made two improvements in her enumeration. First, she organized her counting based on rows of cubes. Second, she started to use counting on strategy instead of counting local groups of cubes separately.

These reattempts at enumeration usually resulted in an improvement in the existing strategy as shown in Episode 1. Sometimes they resulted in a use of a better, more viable strategy as in Episode 2. Sometimes, however, they were detrimental. If the student was not sure about his/her strategy and could not find any justification for the answer, s/he looked for a strategy that would give her/him a plausible answer. In these occasions, students returned to a less viable strategy such as C1 as depicted in Episode 3.
Episode 3: ST is solving Question 3 (graphical). Her final strategy is categorized as C1.

ST: 9.
INT: Can you tell me how you found that?
ST: I counted up here [showing the top layer as she counted the cubes clearly in order of rows] and added one.
INT: Can you show me where the 9th cube is?
ST: [She could not explain her answer clearly and she stopped.]
INT: I want you to explain your answer.
ST: I counted 8 up here, 8 right here and 4 right here [she returned to a different strategy and counted the visible cube faces and found 19, making an addition error].

It should be mentioned, however, that C1 strategy appeared more viable to the students who could not visualize three-dimensionality in drawings of rectangular buildings made of small cubes. Consequently, they easily switched to counting visible cube faces as if they had found a way to find the answer. This return to a less viable strategy was so pervasive that at the beginning of the intervention ST continued to treat cube faces as cubes. She was conceptualizing "the set of cubes in terms of its faces" (Battista & Clements, 1996, p. 263) though she had used A2 strategy for the same 8-cube building in the pre-interview. During the instruction she gradually moved toward a more proper conceptualization of the cube arrays.

In sum, participants usually improved their understanding based on their actions on buildings. However, this was not always the case. Too difficult questions may have forced students to use more primitive strategies than what they were then employing. In a relatively more controlled environment (i.e., instructional period) participants gradually made an absolute improvement.

Assertion 3

*Instruction caused a consistent and long lasting change in students' strategies regardless of the building size and concrete versus pictorial situations. However,*
student's level of reasoning in the numerical domain may have restricted any further development.

Post-interviews were given to each participant six days after the last session of the instruction. While the participants used Category A strategies 12 times in 24 occasions in the pre-interview, after the instruction all of them used Category A strategies for all questions in both concrete and pictorial situations (see Table 3). The consistency of the use of A1 and A2 strategies was evident even for both large size buildings and questions with graphical representations. During the instruction students developed more systematic ways of dealing with cube buildings. After the instruction they usually used A2 strategy, an additive/iterative structure, in 23 out of 24 occasions. For example in the pre-interview DA used C1 strategy to enumerate all the buildings presented pictorially. He used A2 strategy for the same questions including the large one in the post-interview. Figure 2 depicts his strategies on the large building before and after the instruction.

![Figure 2. DA's enumeration strategies before and after the instruction](image)

On one occasion a student (CH) even developed a multiplicative/iterative structure by using A1 strategy for at least a small building as evidenced in Episode 4.

Episode 4: CH is solving Question 2 (graphical). His strategy is labeled A1.
CH: 8.
Int: Can you tell me how you found that?
CH: There is two of these [showing the top layer by circling around it].
The semantics in his expression reflected an emergence of a multiplicative structure. For further evidence, after Question 4 (concrete) the interviewer asked him, "How much is 4 times 9?" Although the question obviously suggested a multiplicative situation for an adult, he answered "9, 18 and 36." He was doing repeated addition instead of multiplication. Although he saw the four layers of nine cubes in the building, he did not multiply 4 and 9 because even 4 times 9 was still addition for him.

Assertion 4

*Changes in student strategies increased student confidence and solution accuracy and decreased the total time to solve the interview questions.*

In the post-interview, except for counting errors in the first layer, not one student who used A2 and A1 strategies changed their strategy after they were asked to explain their solutions. Sometimes they changed their answers but not the strategies. This is evidence that shows that after the instruction the students found A2 and A1 strategies more viable than the other B and C strategies and felt more confident about the results. As seen in Episode 5, while ST was solving Question 4 (graphical) in the post-interview she made a counting error in the first layer. In her second trial right after the interviewer's question, she realized her error and corrected it immediately without any hesitation. She seemed very confident about the strategy she used.

**Episode 5:** ST is solving the question 4 (graphical). Her final strategy is labeled as A2.

ST: 24. There is 6, 6, 6, and 6 [showing front-to-back vertical layers] 6, 12, 18, 24.
Int: Can you tell me how did you find that exactly?
ST: [Recounting the cubes in one layer], there is 9 [smiled], there is nine, there is 9, and there is 9. Nine plus 9 is 18, and 18 plus 18 is 36.
The change in strategies considerably reduced the time (32 minutes in the pre-interview versus 14 minutes in the post-interview, see Table 4) spent on solving problems while it increased the number of total correct solutions. Students in the pre-interview gave 15 correct answers for 24 occasions (mostly small and concrete buildings) while the correct solutions were 100% in the post interview (see Table 4).

Table 4

Students' Speed and Accuracy in Pre- and Post-interviews

<table>
<thead>
<tr>
<th>Student</th>
<th>Time (min) Elapsed</th>
<th>Solution Accuracy*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre</td>
<td>Post</td>
</tr>
<tr>
<td>CH</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>SA</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>ST</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>DA</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Total</td>
<td>32</td>
<td>14</td>
</tr>
</tbody>
</table>

*Note: A total of six questions were asked to each student.

In sum, after the instruction the students increased their speed and accuracy in solving whole array cube enumeration tasks. They usually produced the solutions in one attempt and looked more confident about the results as documented in Episode 6.

Episode 6: SA is solving Question 4 (graphical). She uses an A2 strategy.

SA: 36.
Int: Can you tell me how did you find that number?
SA: There is 12 up here [counting the cubes on the top layer one by one] 12 in the middle and 12 at the bottom. [SA formed the new unit, layer and abstracted the equality of the layers].

Assertion 5

*Counting visible cube faces strategy (C1) is unique to pictorial situations and seems to be related to the perception of three-dimensionality of rectangular arrays of cubes from their drawings.*
A student's strategy was labeled C1 if the student counted only the visible cube faces no more or no less. This strategy never occurred in concrete situations but appeared only in the pre-interview pictorial situations (see Table 4). It appeared that C1 strategy was unique to pictorial situations. It seemed that if the student had not formed or could not visualize the spatial structure of the cube building as three-dimensional from the pictorial representation, he/she used this strategy as a tool to find a number, which might give the number of cubes in the building. For example, DA used Category A strategies for enumerating all the concrete buildings; however, he consistently used C1 strategy for all the questions with graphical representations. He seemed to completely lack any understanding of the three-dimensionality in graphical representations of multi-layer buildings. Similarly, ST used C1 strategy to solve Questions 3 and 4 (graphical). The use of C strategies increased with an increase in stimulus complexity. This also occurred in some occasions during the instructional intervention.

The participants had great difficulty differentiating between the faces of a cube and the whole cube in multi-layer pictorial representations of buildings. Battista and Clements (1996) argued that coordination of different views (such as top, side, and front views of a cube) was a necessary mental operation for this differentiation. However, participants did not have any difficulty with one-layer buildings. A one-layer building consisting of four cubes, for example, has three faces of one cube shown, two faces of a cube shown in two cases (even different faces), and one cube with only one face shown (see Figure 3).
Implicitly or explicitly students could recognize that there were four cubes in the layer. They identified a cube, although only one face was shown, and they did not count the cube in the very front corner three times since it had the three faces shown. This evidence brings up the fact that there are differing levels of coordination (Battista & Clements, 1996) depending on the complexity of the building. Again, concrete and small buildings were less complex and, therefore, easier for students to work with. The following instances further support this claim.

Only one third-grader in the pilot study counted the cube faces instead of cubes in a one-layer pictorial situation. She did, however, count the cubes correctly in a one-layer concrete building. For her, even a one-layer building was complex in a pictorial situation. During the intervention, the interviewer directed her to compare the concrete building to the pictorial one. She first figured out how to find the individual cubes one by one in a one-layer building presented graphically using the concrete building as a referent. The interviewer wanted her to shade one face of a cube, which he was pointing out on the concrete building. After finishing one cube, she was asked to do the same thing.
for the other cubes one by one. While coloring the sides of a cube she might have implicitly coordinated the different views of the cube in a one-layer situation.

The interviewer added another layer to the one-layer building and continued to have her color individual cubes in different colors. Only then she could manage to find and color a row, column, and a layer in the picture. The researcher and the student were able to overcome the difficulty by working together with graphics and concrete materials, and properly comparing and constructing them while systematically increasing the number of cubes in one layer and then increasing the number of layers.

In summary, the complexity of the building was increased with the increase in three dimensions. Therefore, less experienced students like DA and ST, could not attend the three-dimensionality in multi-layer buildings and could not develop proper spatial structuring, especially in graphical situations. As stated by Battista and Clements (1996) the use of all non-layering strategies cannot be attributed to students' misinterpretations of the diagrams. However, the students' difficulties should be accepted in interpreting 2D diagrams as 3D objects, which is a very important mental construction for moving from C to B conceptualization, a topic that is discussed in more detail under "Assertion 7."

Assertion 6

*Initially both numerical and visual clues are used in solving problems but the instruction increased the use of visual (structural) clues.*

All participants in this study said they saw similar, but not the same, cubes and used them before as toys or counters but not for building purposes or for counting
purposes in buildings. All of them had some experiences with Legos. None of them saw the exact drawings of cube buildings.

It was also very obvious from the pilot, intervention, and interviews that the students had very little or no experience with similar materials. They did not even consider what they were doing was mathematics. For them, mathematics meant manipulating numbers with operations such as addition or multiplication. During the pilot study students kept asking, "Are we gonna like solve math problems?" I asked, "Do you think that these are not math problems?" A student replied, "I mean, like real math, addition, multiplication, division."

During the intervention DA's expressions also reflected the lack of these kinds of activities. However, he started to use both numerical and visual information for the tasks. Towards the end, he started asking, "Can I do this like math?" He wanted to write the numbers (number of cubes in each layer) one below the other and do multi-digit addition with paper and pencil instead of mental arithmetic. Obviously, DA was functioning at a low level in numerical domain. Still, he had to find the total of three 12s by using addition algorithms with paper and pencil.

Another student, SA, at first tried to find the number of cubes in the building by dividing it by the number of people who shared the building, and then shaded the cubes she found for each person. Initially her strategy worked well for the small buildings. However, when the size of the building increased she started to have problems with her strategy because finding the total number of cubes was not easy without any systematic enumeration strategy, and she did not have such a strategy at that time. During and after
the activities, she quit her old strategy and started to partition the buildings based on spatial structural elements such as rows, columns, or layers. Pointing students’ attention to the structural elements of cube buildings pulled them away from their old strategies and helped develop new, more systematic approaches to the task.

In general, students such as CH and SA who used relatively more sophisticated strategies for enumerating the cubes in the pre-interview developed very unusual ways of partitioning the cube buildings into equal parts. Two of the examples are shown in Figure 4. Students who used less sophisticated strategies in the pre-interview also moved to the same more viable strategies in the post-interview but they rarely produced that kind of unusual partitioning during the intervention.

![Figure 4](image)

**Figure 4.** Unusual partitioning of the building into four equal parts

Students seemed to get involved with spatial structures of cube buildings in both concrete and pictorial conditions. The kinds of activities used in this study compensated for some of their deficiencies in spatial thinking both in concrete and pictorial situations.
After the instruction, they were able to act on pictorial representations of cube buildings with the same ease with which they worked with concrete representations.

Assertion 7

During the instruction, participants progressed through the three different conceptualizations of rectangular solids composed of small cubes

The three conceptualizations are labeled as C, B, and A type (see Table 5).

Students with a "C" type conceptualization acted on buildings based on faces. Their unit was based on individual cube faces and overall structuring was based on the building's faces. They did not consider the drawing as three-dimensional and did not take the interior cubes into consideration in the concrete building. Students with "B" conceptualization were aware of the three-dimensionality and space-filling properties of the cubes and the whole building. They used cubes as units but their overall structuring was local, not yet global. They usually counted the cubes one by one and unsystematically. It was the "A" conceptualization that enabled students to utilize composite or units of units and unit iteration. For them, the cube building was organized into regular patterns.

Table 5

Students' Conceptualizations of Rectangular Arrays of Cubes

<table>
<thead>
<tr>
<th>Type</th>
<th>Units formed out of</th>
<th>Overall structuring</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>Cube faces</td>
<td>In terms of building faces</td>
</tr>
<tr>
<td>B</td>
<td>Individual cubes</td>
<td>Local</td>
</tr>
<tr>
<td>A</td>
<td>Cubes, columns, layers</td>
<td>Global</td>
</tr>
</tbody>
</table>
All participants comprehended a one-layer building both in concrete and pictorial form before the instruction; however, some of them could not mark a layer (i.e., one of the two shares) on the drawing of a two-layer small building. For example DA and ST were not able to see the equal pieces in the drawings. Instead, they tried to color an equal number of cube faces because their conceptualization of the drawing of rectangular solids was merely a "medley of views" for multi-layer buildings. In other words, their partitioning initially was based on two-dimensional faces rather than three-dimensional spaces.

DA's development through the three types of conceptualizations is depicted in Figure 5. As stated earlier, he completely lacked an understanding of pictorial representations with the exception of one-layered buildings. He first viewed the buildings as consisting of faces –C conceptualization (see Figure 5, view A). Then he vacillated between C and B conceptualization (see Figure 5, view B). His actions while he was coloring the four equal shares were numbered in Figure 5, view B to depict his effort to establish three-dimensional composite units in the picture of a multi-layer building. He first colored the front and upper faces of upper-front row (labeled 1) as one of the four shares. He then colored the front face of the lower-front row (labeled 2) as a second share. After coloring the right side of the lower layer (labeled 3), he recognized that the face (labeled 4) belonged to the lower-front row. At his fifth attempt, DA realized that the face (labeled 5) was part of upper-front row. He finally colored the remaining row (labeled 6) without any hesitation.
As seen in Figure 5, view C, he established three-dimensionality in a pictorial representation but could not go beyond local structuring. Although it looks like a layer-type structuring it is not because it is supposed to be partitioned into four equal pieces instead of three equal pieces. Finally, towards the end of instruction he reached an "A" conceptualization even with a large building as seen in Figure 5, view D.

![Image of three-dimensional structures with labels A, B, C, and D.]

**Figure 5.** DA’s development through the three types of conceptualizations

Another student, SA, went through similar stages. Although she used A2 strategies for enumerating the small buildings, she returned to the C3 strategy for the large building. At the initial stages of the intervention she was not flexible enough to partition the buildings, even the small ones, into a different number of shares. Therefore, she approached the task as "a set of cubes in terms of its faces" (see Figure 6, view A). With encouragement from the interviewer, she established units in the picture based on
three dimensions and started to use "local structuring" (see Figure 6, view B). From the small buildings, it was not possible to determine whether her conceptualization was A or B. Both could be possible. However, when the size of the building increased it became obvious that her conceptualization was a B type (see Figure 6, views B and C).

![Figure 6. SA's development from C to B conceptualization for the small and medium building](image)

There was another reason why she used local structuring. She was relying more on a numerical strategy and she was not able to attend to the spatial properties of the whole building. As described in Assertion 6, she initially in her mind found the total number of cubes in the building and then divided it by the number of sharers. She then colored that number of cubes. If she obtained the correct total number correct, her partitioning was accurate. However, her developing B conceptualization was so fragile that she returned to C conceptualization because of the increasing size of the building (see Figure 7, view A). After awhile, with the help of concrete materials she established the B conceptualization for the large building. However, her partitioning was still not regular (i.e., not based on layers; see Figure 7, view B). Her final strategies towards the end of the instruction were clearly a layering type (see Figure 7, view C). Her actions
with the problems that encouraged equal partitioning of the cube buildings helped her to construct the regular spatial patterns in the buildings. SA's three-step development represented with her drawing tasks is depicted in Figure 7.

![Figure 7](image)

**Figure 7.** SA's development through the three types of conceptualizations

It can be observed from her colorings (see Figure 7) that, again, for the large building SA started with a C conceptualization. Then she managed to move to the B conceptualization. With some guidance from the interviewer and using concrete materials, she was able to use the "A" type conceptualization by systematically and simultaneously acting on both concrete and pictorial representations of the cube arrays.
However, even the partitioning of concrete buildings was not easy for the participants at the beginning. They were usually able to construct the equal shares on small concrete buildings. However they were not able right away to do the same with the drawings. The consistent checking of the equality between concrete blocks and their drawings brought them to a point where they were able to abstract the equality of the structural elements of the rectangular cube buildings such as rows, columns, and layers. Gradually, they could manage to apply this strategy to both large size and pictorial buildings.

Assertion 8

*During the intervention, three types of interaction contributed to the invention of new strategies.*

Three types of interaction contributed to the students' understanding. One was the interaction between students. Second was the interaction between the students and the interviewer. Third was the interaction of the students with the task and the material.

Although their classroom teacher described the students CH, DA, and SA as below average and ST as close to average in mathematics before the interviews, their performance in cube enumeration tasks during the pre-interview did not correspond with this categorization. In each group of pairs (CH & DA and SA & ST), CH and SA looked more competent than the other partners in solving cube enumeration tasks. Subsequent interactions during the instruction between the partners showed that each partner benefited from the other, though the less competent gained more at the initial stages of the intervention. The social interaction between the partners afforded an environment in which one learned from the other. This interaction was not unidirectional. That is, in
some situations one was more active and explanatory and in some other situations the other was more verbal. Both learned from each other through negotiation.

In the pre-interview and during the initial stages of the intervention, ST was not able to distinguish between cubes and cube faces in drawings of multi-layer rectangular buildings. Although she used A2 strategy to enumerate the small building (drawing of a 2x2x2-cube building), she returned to C1 strategy for the larger two buildings. At the very beginning of the intervention, she confused cubes with their faces even in the same small building. Her partner explained that the three adjacent faces (top, front, and right side) belong to one cube. She was struggling to see what her partner was explaining to her. After awhile she started to construct individual unit cubes in the drawing and internalized this behavior. Later in the instruction she even discovered different equal amounts that seemed quite sophisticated. Episode 7 illustrates her transition from "the set of cubes in terms of its faces" to "the rectangular array organized into layers" type of conceptualization for the small size building.

Episode 7: A 2x2x2 building to be shared between 2 people.
01. Int: You have this building and you want to share it equally between two people. How would you do that?
02. SA: There is 8 squares.
03. ST: Four for you [showing the top side] and four for me [showing the front side] and then two for you [showing the upper row on the right side] and two for me [showing the lower row on the right side. She was dealing with the cube faces].
04. SA: No, but there is 8 squares [meaning cubes] in whole. So, each gets 4.
05. ST: There is 12 [counting the cube faces visible on the picture].
06. SA: They are gonna be sides.
07. ST: No, look this is two [touching the top side of the two cubes in the top layer].
08. SA: They are just tops.
09. ST: [Looks at the drawing] Yeah [looked not convinced].
10. SA: See, there is two halves [She was functioning at the level of A conceptualization at least for this small building]. This half is 4. There is 8 squares [meaning cubes]. So, each gets 4, right?
11. ST: Uh-um [looked still not convinced].
12. Int: Are you agreed on this?
13. ST: Look, look, there is two right here, and two right here [showing the faces visible on the top side of drawing].
14. SA: They are top of the square [meaning cube].
15. ST: There is two right here [showing the cube faces in the upper row of the front side].
16. SA: This is side, top, and side of square [showing the three faces of the cube on the corner].
17. ST: There is two right here [showing the upper row in the front side].
18. SA: This is the top of the square [meaning cube]. [SA did not realize that ST started to see what she was explaining. She meant the upper front row. As if ST was starting to see the drawing as a 3-dimensional object as she was indicating the row consisting of two cubes].
20. ST: See, there is two right here [again tracing the boundary of the cubes in the upper front row] and two right here [showing the bottom row in the front] and one here [pointing the cube at the bottom right end] and there is another you can't see.
21. SA: OK, you see this line dividing the two halves [showing the middle vertical line on the right side]. This is four squares [cubes] over here. This is the side of the square [showing one face of a cube].
22. Int: Cubes, let say cubes.
23. SA: Right, cubes. This is the side of the cube and this is the side of the other cube [explaining the picture]. This is one cube and 8 cubes in all. And we want to share right in the middle and each gets four.
24. Int: Do you agree with her [to ST]? 
25. ST: Uh um.
26. Int: Can you shade the two equal parts in two different colors? 
27. SA: What do you mean?
28. Int: Can you shade one share in one color and one share in another color? 
29. ST: Can I just 4 here and 4 here? 
30. Int: OK.
31. ST: So, I color my share and she colors her share? 
32. Int: Yes.
33. ST: OK, I colored (Figure 8).

Figure 8. ST's shading of a one share.

As depicted in Episode 7, ST's conceptualization of "the set of cubes in terms of its faces" is apparent and lasted until line 13. But at the same time she was negotiating with her partner about the meaning based on their interpretations of the building. In line
15, she showed some evidence that she was able to see the two cubes in the upper front row. They continued negotiating. In line 20, she established the three-dimensionality and visualized at least some of the eight cubes in the small building. The interviewer helped them use the correct word for what they were negotiating. In line 29, she was able to visualize the two layers consisting of four cubes, each through halving and applied her knowledge on the drawing as seen in Figure 8.

Since ST knew already how a cube and a one-layer building looked like in a drawing (because she correctly counted the cubes in both situations) she managed to see the individual cube and one individual layer in the small building and separate it from the rest, as if she determined a landmark (which she knew before) in the scene (Battista, 1994). Her interactions with a peer, interviewer, and the material all helped her construct this understanding, for the time being, in a small building.

Similar instances occurred during CH and DA's partnership. Compared to ST, DA seemed less competent in interpreting the pictorial representations of cube buildings. In the pre-interview, he used C1 strategy in all the pictorial situations except for the one-layer building while ST used C1 strategy for only medium and large buildings. DA's inability to act on pictorial representations was also evident during the initial stages of the intervention. He was not sure what to do on the drawing. Therefore, it took him awhile to transition from "the set of cubes in terms of its faces" to "the rectangular array organized into layers" type of conceptualization of the rectangular buildings made of small cubes. However, the positive influence of all the interactions (i.e., peer, interviewer, and material) was in effect during the instructional period as illustrated in Episode 8.
Episode 8: A 2x2x2 building to be shared between 2 people.

Int: Let's read the question first. You have this building [showing the 2x2x2 building] and you want to share it equally between two people. How would you do that? Do you understand the question?

CH: Yeah. You want me to do?

Int: Please.

CH: I give one person this one [showing the 2x2x1 vertical layer] and one person this one [showing the other layer].

Int: DA, is your solution same?

DA: Yeah, same.

Int: OK, I want you to shade one share with a color, just one share. Leave the other share uncolored.

CH: [Made the coloring very quickly as in Figure 9, view A].

DA: [Colored just the front face first (see Figure 9, view B), then asked if he needed to color the sides].

Int: See, the question is two people sharing this building, so shade only one share. You said 4 cubes for each person, right? Can you shade the four cubes only?

DA: [Looked puzzled and hesitated]

Int: You shaded only the face not the whole cubes.

DA: Do I shade the whole [started to shade the entire building (see Figure 9, view C)]?

CH: No, not the whole thing. Look, you have to do that and you have to do that [showing the top and the right side of the front layer in DA's drawing].

DA: [Colored the faces of front layer, with no indication of understanding].

Figure 9. CH and DA’s shading of one share on the drawing of a 2x2x2 building

Int: Do you know why? Why we shaded this and this too [showing the top and the right side of the front layer].

DA: To make it equal.

Int: That's right, to make it equal, what else?

CH: So you can see the 3D in it.

Int: OK, because it is 3D.

CH: So you can see altogether in it.

Int: DA, do you know what a 3D is?

DA: Yeah, like, like, like. It's like pops up. [He doesn't seem to have the understanding that a 3D occupies a space; rather he thinks that a 3D consists of an organization of individual faces].

Int: OK, can you see the 3D in this picture [showing the picture he already colored].

DA: No.
CH: I can.
Int: Can you see the 3D in this [showing an uncolored drawing of a 2x2x2 cube-building]?
DA: Yeah.
Int: You couldn't see in this because you colored it too dark [joking]. Let's make the building with cubes. You make one share and you make one share. Put them together and make the whole building.
Both: [Each made one share, a 2x2x1 vertical layer, out of cubes].
Int: DA, can you see the 3D right now?
CH: Yes. This is 3D [showing the cubes]. He is 3D. You are 3D.
Int: OK, let's put them together. Now, think about how many ways of sharing this building between 2 people you can find.
CH: Yes [makes two different ways of vertical layering].
DA: [Separates the top and bottom layers first, then left and right layers and thinks about it for awhile and makes the front and back vertical layering with concrete cubes. He doesn't have any difficulties with small concrete buildings but with pictorial and/or large buildings as evidenced in his enumeration tasks].

As seen in Episode 8, CH is acting as a stimulator for DA to improve his understanding by explaining to him the three-dimensionality of the building. Although DA seemed to understand the problem and its solution, his coloring task showed that he was still confused between cubes and faces. After working with concrete material (see Figure 10) he was able to complete the coloring task by comparing the shares in the concrete building to the ones in the drawing.

The coloring task made it obvious that DA had some difficulty acting on pictorial representations of rectangular solids. He was not able to construct unit cubes on the drawing based on three-dimensions. The researcher was able to see that he did not have any difficulty doing the sharing tasks with the concrete materials as shown in Figure 10. So the problem he had was not about the sharing task but the pictorial situation. DA successfully transferred his understanding to pictorial situations through constructing and comparing concrete buildings with graphics.
The inquiry between the interviewer and the participants seemed to be stimulating for the students' construction of equal pieces in the building. Appropriate teacher guidance enabled students to re-think about and look at the problem situation from a different angle. These new approaches to the problems by the students, in some situations, might have initiated some kind of mental action that led them to a higher-order of thinking about the problem at hand. In Episode 9, for example, successive questioning by the interviewer of DA by having him do the mathematical operations led DA to integrate the operations and form a new level of thinking. As a result, he does not have to count all the cubes in the building one by one. Instead, he could do addition since he had already abstracted the equality of the layers. On the other hand, although questioning suggests a multiplicative situation, DA used addition since his numerical development was only at this level.

Episode 9: DA is working on sharing a 4x3x3 building between 3 people.
Int: DA, you have three layers, right?
DA: Uh-um.
Int: How many cubes in one layer?
DA: 12 [counting the cubes with eye movements].
Int: How many do you think altogether?
DA: I think [waits]. Can I do it like a math problem?
Summary of Results

Before the instruction participants exhibited three different conceptualizations of rectangular arrays of cubes. Their conceptualizations were inferred from the strategies they used in dealing with cube enumeration problems involving both concrete and pictorial representations of the buildings. Before the instruction the complexity of the task, as explained with the size of the building and the mode of presentation, affected their strategy choices. During the instruction they developed more viable strategies in dealing with cube buildings. After the instruction, all the participants used layering type strategies regardless of the complexity of the buildings in cube enumeration tasks.

Equal-sharing situations afforded students to develop spatial and numerical composite units leading them to use more systematic enumeration strategies. However, understanding three-dimensionality in a picture was a prerequisite. They already established three-dimensionality on the drawings of individual cubes and one-layer buildings before the instruction and they knew what a cube was. However, their understanding was not established well enough to carry the similar task in multi-layer buildings.

During the instruction participants improved their approach to the tasks and moved to a higher level of understanding of the rectangular solids made of small cubes both in spatial and numerical domains. The instructional tasks used in this study engaged students in different levels of unitizing in both geometric and numerical domains, each to
the benefit of the other. Then, they developed the additive/iterative and multiplicative/iterative structures to find the total number of cubes in rectangular buildings instead of trying to count all one by one.
The purpose of the present study was to reveal whether the activities provided in the context of equal sharing that emphasized both numerical and spatial aspects of cube configurations caused any improvement in students' strategies that led them to form composite units and iterate them in enumerating the cubes contained in rectangular buildings. A three-step procedure was followed in order to accomplish this purpose. At first, the students' level of reasoning with enumeration tasks was determined. In the second, they were provided with a series of activities. Finally, their level of reasoning with the same task was reassessed. The eight assertions presented in Chapter 4 are discussed under the following three sections: "The Three Conceptualizations," "Task Complexity," and "Instructional Effect."

The Three Conceptualizations

The participants in this study exhibited three different conceptualizations of rectangular solids made of small cubes. This finding is consistent with the spatial structuring theory (Battista & Clements, 1996). Additionally, in each level of conceptualization there were two related processes to be accomplished. One was the formation of units. What the students formed as a unit was inferred from what the students were counting while enumerating the cubes. They may have counted cube faces, individual cubes, and some composites of cubes.

The other was the overall structuring or organization of these units in the whole building. How the students structured the building was inferred from how they were
enumerating the units. The possibilities were counting units with reference to building faces, counting unit cubes in local groups, and counting in regular patterns such as layers.

In the first level, students viewed the buildings as a "set of cubes in terms of its faces" (C conceptualization). Since typical "C" conceptualizers could attend only one face at a time (i.e., no coordination of views) they took the cube faces as units to be counted and organized them in terms of building faces. They did not bother with invisible cubes or even faces.

Second, they conceptualized the "set of cubes as space filling" but did not utilize layers. Their structuring was based on unit cubes but it was local, not yet global (B conceptualization). Typical "B" conceptualizers were aware of the space filling and three-dimensionality properties of cubes and the building itself. As a result they tried to count all the cubes inside and out but they could not see any organization of the cubes in the building. So after they first tried to count cubes consecutively (one by one), they then developed skip counting. However, the skipped number did not correspond to any regular pattern, such as columns and layers in the building. Therefore, their overall structuring was considered local.

Third, students conceptualized the "set of cubes organized into layers" (A conceptualization). Typical "A" conceptualizers started to properly iterate units based on spatial structural elements of the buildings. They could flexibly view the buildings and form regular patterns such as columns and layers in the building. The global structuring of the whole building was achieved. Skip counting, additive, and multiplicative iterations were successively inserted into their enumeration strategies.
In sum, the difference between C and B conceptualization is the utilization of three-dimensionality or forming proper units based on three dimensions of individual cubes in multi-layer buildings. The difference between B and A conceptualization is the organization of the structural elements of buildings in a systematic way or forming units of units or composite units and then unit iteration.

If this is so, why is it that the same students used different conceptualizations for different tasks? There are two reasons for this. One is the complexity of the buildings such as concrete versus pictorial and small versus large buildings. The other is the need for a systematic experience. These will be discussed in the two subsequent sections.

Task Complexity

In pre-interviews and instructional periods, students were not consistent in their use of strategies from which their conceptualizations were inferred due to "the task complexity" (Middleton & Corbett, 1998, p. 263). The strategies they used in dealing with cube buildings were distracted by both the increasing sizes of the buildings and pictorial situations.

It was relatively easier for students to visualize a small building (i.e., a building containing a fewer number of cubes) than a large building. That was why a building increasing in size forced students to use more primitive strategies. Additionally, it was not obvious from the small buildings if the students had a local structuring or a layering-type structuring since the buildings were too small to determine this difference. Later, it became clear from the large buildings that their structuring was local; or it could be that they just started to form the layering-type structuring for small buildings; however, their
conceptualization was so fragile that it could easily be distracted by the increasing size of the building.

Pictorial situation was another effect that disturbed the students' spatial structuring. The reason for this appears to be the students' difficulty in visualizing the three-dimensionality in pictorial representations. Before the instruction, all the participants used less viable strategies for the problems presented pictorially. In addition, students (especially C conceptualizers) used completely different strategies for concrete and pictorial representations.

For students, it was relatively easier to perceive three-dimensionality in concrete representations than it was from pictorial representations. Therefore, a student might use a B conceptualization for a concrete building, while s/he was still utilizing a C conceptualization for a pictorial representation of the same building. The reason was her or his difficulty in viewing the three-dimensional units in pictorial representation.

Similarly, Battista and Clements (1996) found that students shifted from C to B strategies as they moved from the problem presented pictorially to the ones presented concretely.

Lacking a systematic experience, students' proper structuring is suppressed. The participants of this study had very little experience with similar concrete materials and no experience at all with similar pictorial representations. Similar findings are reported in the literature (e.g., Ben-Chaim et al., 1985). Without a direct experience with this type of drawing most of the students cannot visualize three-dimensionality from the drawings. It needs considerable conventionalizing (Bishop, 1979). They need to be guided, supported,
and instructed in "an appropriately structured environment" (Koedinger, 1998, p. 319) like the one used in this study.

The Effect of Instruction

Without the instruction students usually used the kind of strategies with which they tried to account for either faces of cubes or individual cubes rather than three-dimensional composite units such as columns and layers. During the instruction that emphasized both numerical and spatial properties of cube buildings in equal sharing situations that were provided with concrete and pictorial representations, they progressed through the three types of conceptualizations. After the instruction, students became more consistent in their use of layering strategies regardless of the size and mode of presentation. How did the students develop from one state of viewing to the other?

In the formation of units based on three-dimensions (from C to B conceptualization), one should attempt to coordinate and integrate relevant views of that unit in order to make a mental model of the unit (Battista & Clements, 1996). Participants in this study did these operations by determining and coloring individual cubes in the drawings of multi-layer buildings by looking at the concrete buildings. Since they had already formed the three-dimensional units (i.e., cubes) on small size concrete buildings and on the drawing of single cube and one-layer buildings they could transfer and integrate this knowledge within the context of multi-layer buildings.

An "A" conceptualization requires a higher level of abstraction, such as the equality of layers and global structuring of cubes into composites. For example, after
counting the cubes in one layer, a student should count the layers as if they are units. Then, s/he can use either skip counting or addition or multiplication to find the whole.

In the formation of composite units based on three-dimensional units, students should engage in the iteration of units of units. Participants in this study experienced the equality of layers on concrete buildings and reimplemented their ideas on the drawings of buildings. Eventually, they assumed the equality of layers although they did not seem visually equal in the drawing due to occlusion. For the same reason, they did not have the chance to check if the layers were equal on the drawing. On the other hand, they had many ways of checking the equality of layers in concrete buildings. Participants first did the partitioning on the concrete building; then they transferred this information to pictorial buildings.

The hypothesized processes in this research are further supported by the fact that a systematic instruction caused a consistent and long lasting change in student strategies in dealing with rectangular buildings made of small cubes. During the course of the instruction, the three types of conceptualization were also observed on students' work. They moved from one state of viewing to the other as a result of observable interactions with peers, interviewer, and the task materials.

Peer interaction was helpful in situations where one of the partners developed a better understanding and tried to explain it to the other. The less developed partner imitated her/his partner for awhile but finally reached a better understanding through her/his own actions and/or observations of the other. The effect of peer interaction was paramount especially at the initial stages of this development. Through imitation and then
negotiation, peers modified their understanding of the three-dimensionality in general and rectangular solids made of small cubes in particular.

Students' interactions with the interviewer were also influential in situations, where the students needed proper and prompt questioning. While trying to find an answer to the interviewer's questions, participants developed a better understanding of the problems at hand and utilized a more sophisticated strategy. For example, while they were using "counting on" strategy to find the number of cubes in successive layers in the building they invented additive/iterative strategies through some verbal or perceptual inputs from the interviewer who provided input through questioning cycles.

Another positive effect came from the interaction that took place between the participants and the instructional material designed and used for this study. Systematically constructing and deconstructing rectangular buildings demanded by equal sharing and construction situations, comparing the concrete buildings with their pictorial representations, and students' coloring actions on pictures brought about tremendous changes in students' spatial and mathematical thinking. Students abstracted the equality of layers and exhibited this thinking in their actions. For example, they did not count the cubes one by one for each layer but instead counted the number of cubes in one layer and iterated that number such times that was equal to the number of layers in the building. In sum, in the act of creating patterns, students built relationships that transcend geometry and involve mathematics in general (Wheatley, 1992). For example, they started with "counting on" strategy, which is treating a layer as an iterable composite unit (Battista & Clements, 1996), and continued skip counting, addition, and multiplication, in order of
which shows respectively that the student became more sophisticated in the numerical domain, as well.

During the instruction, students customized the idea that a building consisted of layers and layers consisted of unit cubes. They explicitly stated that layers are equal to each other by saying, "There is two of them," or "one right here, one right here, and one right here." This idea of unitizing (i.e., taking a layer as a unit) led them to develop the additive and multiplicative/iterative structure in three-dimensional spatial constructions. The "unitizing" operation is a general mental operation that is fundamental to many mathematical concepts (Wheatley, 1992). The activity of constructing units is a mental operation of segmenting experiences, isolating one aspect of experience while at the same time leaving it embedded in the whole (Clements et al., 1997).

As stated earlier, students had difficulties acting on pictorial representations at the initial stages of the instruction. However, properly returning to a simpler situation, having perceptual inputs from the concrete buildings, and constantly comparing different solutions helped the students overcome their difficulties. At the end of the instruction, there was no such difficulty. They implicitly coordinated and integrated different views of cubes and cube buildings through their coloring actions.

Coordination and integration of views is a necessary process to establish three-dimensionality in the picture as well as on concrete buildings. However, it is a long process that needs time and appropriate activity to be established (Battista & Clements, 1998). Once it is in place is not a guarantee that the student can do it for any stimulus. It still depends on the complexity of the target object. In this case complexity increased
with the size of the building and pictorial representations appeared to be more complex than their concrete counterparts.

The coordination and integration operations were hindered by the increasing size of the building. This caused a lack of understanding of three-dimensionality in the drawings of multi-layer cube buildings. After the instruction both of these detrimental effects were gone. A plausible explanation might be that the students developed proper mental images that gave them the flexibility to use more efficient and systematic strategies to deal with the enumeration tasks by establishing proper units and units of units such as layers from the structural elements of the cube buildings. Starting from the tasks with small buildings they formed their mental images by implicitly coordinating and integrating views of cubes and cube buildings through their coloring actions.

This claim is further supported by the finding that none of the students had encountered any difficulty with the one-layer building but multi-layer buildings before the instruction. After they formed a global mental picture of the whole building they looked for the equal pieces such as layers that they formed previously. This enabled them to use the strategies based on layers. Layers acted as the basic building bricks of this construction. This claim is consonant with one of the canons of the spatial structuring theory (Battista & Clements, 1996).

Students came to this understanding by acting systematically and simultaneously on the concrete and graphical representations of cube buildings. The choice of tasks and operations such as gradually increasing the size of the cube buildings and coloring the equal parts paved the road to the construction of necessary mental images and structures.
Returning to a simpler situation in cases where the task appeared to be overwhelming to students afforded a remediation and compensation. For example, students returned to building cubes whenever they got stuck with a graphical situation.

Theoretical implications, implications for future research and educational implications are discussed in the subsequent sections.

Theoretical Implications

Children's overt actions in performing a spatial task can be understood as expressions of a set of underlying constructive mental acts (Wheatley & Cobb, 1990). Students' conceptualizations of rectangular buildings made of small cubes can be inferred from their actions on these buildings and hierarchically categorized into three different types. For type C, students structure the building based on two-dimensional elements such as cube faces or building faces. For type B, students accept the cubes in the building as space filling objects but cannot form proper composite units; their structuring is local. For type A, Students can visualize the building as consisted of regular patterns, such as layers. They can flexibly act on buildings to enumerate and/or to partition them into different numbers of shares.

There are two related processes within each level of conceptualization: formation of units and overall structuring. What the students are enumerating tells about the units they formed. How they are enumerating is the indicator of their overall structuring of the whole building. A "C" type counts cube faces one by one organized into visible building faces. A "B" type counts the cubes one by one inside and out, then continues to count cubes in local organizations. An "A" type counts the cubes in one layer and then iterates
the layer along the third dimension using either skip counting, additive, or multiplicative schemas.

The transitions from one state of viewing to the other can also be observed from their actions on rectangular buildings made of small cubes. During these transitions they exhibit a mixed-type conceptualization. Every time they are presented a larger building, for example, they return to a primitive conceptualization or a mixed-type until they have established well the "A type" conceptualization in multi-layer buildings. Therefore, the current study additionally coined the term "complexity" of the task that affected the students' struggle to structure rectangular buildings made of small cubes.

Complexity increases in two dimensions. One is the size of the building and the other is the mode of presentation. That is, the larger the building is the more complex it becomes, and pictorial representations are more complex than their concrete counterparts. During the course of the development from C to A conceptualization, students benefited from increasing and decreasing the complexity level whenever needed. They built on the understanding they gained from the simpler situation and gradually improved their level of functioning.

The development of the students' understanding of rectangular solids made of small cubes is not a matter of solely spatial development, however. Rather, this development is dependent on their level of numerical reasoning. In order to develop further students needed more sophisticated numerical reasoning. Similarly, Wheatley and Reynolds (1996) found a consistent parallel between students' sophistication of the types of units constructed in a geometric setting with their numeric activity.
There is evidence in the current study, for example SA and CH, that shows that students’ prior numerical reasoning (gained without a spatial experience) hindered their spatial development. After a systematic instruction, however, all students, including SA and CH, started to use both visual and numerical clues to solve the cube enumeration tasks although they mostly used numerical strategies before the instruction. This development gave them more speed, accuracy, and the flexibility to act on rectangular buildings made of small cubes. Additionally, the use of visual clues stimulated their mathematical thinking about unitizing. For example, they formed proper units, units of units, and finally abstracted the equality of composite units by iterating them as units.

Implications for Future Research

The present study focused on a small portion of the problem. Therefore, some relevant questions still remained unanswered. First, it would be beneficial to add a task in which the students are given only the dimensions of rectangular array to be partitioned. This assessment would provide information about how the students function on those tasks that have relatively less perceptual input.

Second, the kind of activities used in the present study helped the students to structure cube arrays in terms of layers. However, students found the number of cubes in one layer by counting them one by one. There is certainly a need to emphasize the additive and multiplicative structure in two-dimensional arrays in order to help students improve their strategies toward a better understanding of the measurement of area and volume. Future research should aim at improving 2D structuring too.
In the present study, students rarely (if any) used multiplication to enumerate the cubes contained in rectangular arrays, even when the situation clearly suggested multiplication to an adult. A possible reason for this could be the students' low level of reasoning in the numerical domain (all the participants were below average in mathematics). Future research should take the average and above average students to see if they behave differently.

Educational Implications

Finding the number of cubes in rectangular buildings made of small cubes was indicated one of the problematic areas in mathematics education. Students' performance on these kinds of tasks depends on their spatial structuring of the cube arrays (Battista & Clements, 1996). In this study, students' spatial structuring was stimulated by providing them a rich environment in which they could act on materials and interact with others. As a result they moved from a poor to a more appropriate and richer conceptualization of the 3D arrays of cubes.

Whole array enumeration is a long constructive process. Based on the evidence gathered through this and previous research it was hypothesized that if the task were broken the task into comprehensible steps then the student could make the necessary transition. To do this, diagnosing students’ existing knowledge and current level of functioning is an important task to start with. The participants in this study did not have any difficulty with one-layer buildings either concrete or pictorial. Additionally, they had some understanding of the small building (i.e., a 2x2x2 building). Therefore, it seems
reasonable to assume that this accumulation could be used as a stepping stone towards more complex tasks while keeping track of the students' understanding.

An instructional model has been proposed in accordance with the results. The following is mainly a three-step instructional model that worked for these four 4\textsuperscript{th} graders.

1. \textit{Assessing the student level of functioning}: If the student is not ready to develop a structure, all of the explanations and reinforcements that might be tried will not increase understanding (Phillips et al., 1994). Therefore, students should be assessed with a range of possible situations in order to find out the flexibility and strength of their thinking. Buildings that have a small number of cubes may be misleading in determining the student's current level of thinking. Both concrete and pictorial representations in separate settings should be used to see if the student is functioning differently on them. A possible chain of questions is provided in Appendix B. Evaluate the students' strategies using the table provided in Appendix A.

2. \textit{Providing appropriate instruction}: Based on the results gathered from step 1, assign students activities contained in Appendix C using the instructional model in Figure 11.

In the present study, students with C conceptualization benefited from the kind of instruction in which the instructor guided them to determine individual cubes and then color their faces so as to mark them differently on the drawing while the instructor was pointing out the cubes and faces on the concrete building. These actions helped them to
form units based on the three dimensions of cubes. They also experienced the extreme cases in which only one face of the cube was shown.

Students with B conceptualization benefited from the kind of instruction in which the instructor guided them to build the concrete building by looking at the drawing. Then, they partitioned the concrete building into some numbers (i.e., the number of layers in one of the dimensions) and implemented this partitioning on the drawing. Partitioning in all three dimensions was also encouraged by asking the students to find other possible ways of sharing the concrete building and implementing these solutions on drawings of buildings.

Students in transition from B to A conceptualization seemed to be benefited from the kind of instruction in which they were encouraged to find different ways of partitioning the concrete building and implementing these solutions on the drawings, and afterwards checking for their correctness. Students at all levels needed and benefited from the kind of practice in which the complexity of the task was being increased and sometimes decreased.

Students with a well-established A conceptualization both on concrete and pictorial buildings could easily act on buildings to partition them into different numbers of shares. They are also ready to enumerate the buildings using layer iteration strategies. Towards the end of the instruction all the participants exhibited this behavior. In the post-interview all of them used A1 and A2 strategies to enumerate the buildings as expected.
3. **Assessing the improvement:** Reasses the students with pre-interview questions to see if they made any improvements as a result of instruction. Evaluate students' strategies using the table in Appendix A. Compensate if necessary.

![Instructional activity flowchart](image)

*Figure 11. Instructional activity flowchart
(*X could be the number of rows, columns for small buildings, and layers for the large building*
REFERENCES


APPENDIX A

STUDENTS’ STRATEGIES WHILE ENUMERATING CUBE ARRAYS
Students' Strategies While Enumerating Cubes Arrays

A. "The student conceptualizes the set of cubes as forming a rectangular array organized into layers
1. Layer multiplying: Student computes or counts the number of cubes in one layer (vertical or horizontal) and multiplies by the number of layers.
2. Layer adding/ iteration: Student computes the number of cubes in one layer (vertical or horizontal) and uses addition or skip counting (pointing to successive layers) to get the total.
3. Counting sub units of layers: Student's counting of cubes is organized in layers, but the student counts by ones or skip counts by a number that does not equal the number of cubes in a layer. For example, the student counts the top layer by ones, then counts on from the result, again pointing to each cube in the top layer, for each of the two remaining layers.

B. The student conceptualizes the set of cubes as space filling but does not utilize layers.
1. Column/row iteration: Student counts the number of cubes in one row or column and uses skip counting (pointing to successive rows or columns) to get the total.
2. Counting sub units of columns or rows: Student's counting of cubes is organized by row or column, but the student counts by ones or skip counts by a number that does not equal the number of cubes in a row or column. For example, the student counts by twos or ones, pointing successively to columns of four.
3. Systematic counting: Student counts cubes systematically, attempting to count both inside and outside cubes. He/she might, for instance, count the cubes on all the outside faces, then attempt to determine how many are in the center.*
4. Unsystematic counting: Student counts cubes in a random manner, often omitting or double counting cubes, but clearly tries to account for inside cubes.*

C. The student conceptualizes the set of cubes in terms of its faces.
1. Counting subset of visible cubes: Student counts all, or a subset of, cubes on the front, right side, and top-those that are visible in the picture.*
2. Counting all outside cubes: Student counts outside cubes on all six faces of the prism.*
3. Counting some outside cubes: Student counts outside cubes on some visible and some hidden faces but does not count cubes on all six faces of the prism.*
4. Counting front layer cubes: Student counts outside cubes in front layer.
5. Counting outside cubes, but not organized by faces.
D. The student uses the formula L x W x H.
Student explicitly says he/she is using formula, or implies it by saying, "Multiply this times this times this" (pointing to relevant dimensions). There is no indication of understanding in terms of layers. (If students used the formula, they were asked, "Why did you multiply these numbers together? Why does this work?")
E. Other.
Student uses a strategy other than those described in A-D, such as multiplying the number of squares on one-face times the number on another face.

*This strategy was used, and cubes on some edges were double counted" (Battista & Clements, 1996, p. 263).
APPENDIX B

CUBE ENUMERATION TASKS
CUBE ENUMERATION TASK
This is a small (unit) cube [showing the actual wooden cube]. And this is how we draw it

[showing this drawing].

This is a rectangular building made up of small cubes [showing each building below]. How many small (unit) cubes will it take to make this building? The buildings are completely filled with cubes, with no gaps inside.

1. [Diagram]
2. [Diagram]
3. [Diagram]
4. [Diagram]
APPENDIX C

INSTRUCTIONAL ACTIVITIES
INSTRUCTIONAL ACTIVITIES

Activity 1: You have this building [showing the drawing of 2x2x2 building]. You want to share it equally between (2) people. How would you do that? Can you show me the equal parts by shading them in different colors on the drawing? How do you know that you shared equally? Can you check with concrete building? Is there any other way to share it equally?

Activity 2: A 2x2x2 building to be shared between 4 people.

Activity 3: A 3x2x2 building to be shared between 3 people.

Activity 4: A 3x2x2 building to be shared between 4 people.

Activity 5: A 2x2x4 building to be shared between 4 people.

Activity 6: A 4x3x2 building to be shared between 3 people.

Activity 7: A 4x3x2 building to be shared between 4 people.

Activity 8: A 4x3x3 building to be shared between 3 people.

Activity 9: A 4x3x3 building to be shared between 4 people.

Activity 10: A 4x4x3 building to be shared between 3 people.

Activity 11: A 4x4x3 building to be shared between 4 people.
APPENDIX D

VERBATIM TRANSCRIPTIONS OF THE DATA
DA is a fourth grader and is described by his teacher as a below average student in mathematics. He has never seen such cubes before but seen larger ones and used them for counters. He also played a little bit with the Legos.

GQ1. - 4 [just counted the cubes one by one without any hesitation]

GQ2. - 12 [counted the cube faces seen on the three side of the prism]

GQ3. - 20 [counted the cube faces seen on the three side of the prism]

GQ4. - 33 [counted the cube faces seen on the three side of the prism as seen below]

GQ5. - 12 [he was having difficulty seeing the drawing as a 3D box]

CQ1. - 9. Six here [showing the two rows together] and 3 more, 7, 8, 9, 10, wait [started over] 1, 2, ... 9 [counted individual cubes one by one in order of rows]

CQ2. - 8, 9 [as she moved around the building].

CQ3. - 16

CQ4. - 36

CQ5. - 12 [he was having difficulty seeing the drawing as a 3D box]

CQ6. - Can you tell me how you did that?

- Just added them [he added the three number to find something as an answer].

CQ7. - 9. Six here [showing the two rows together] and 3 more, 7, 8, 9, 10, wait [started over] 1, 2, ... 9 [counted individual cubes one by one in order of rows]

CQ8. - 8, 9 [as she moved around the building].

CQ9. - Which one do you think is correct?

- 8

CQ10. - Can you show me how you got that?

- There is 4 here and 4 here [showing the two vertical layers]. *[He started to count the cubes all around unsystematically and found 8 and 9. Then, in his third attempt he organized his counting into A2 strategy]

CQ11. - 16

CQ12. - Can you show me how you found it?

- I just counted this side twice [showing the front layer]

CQ13. - Why did you do that? I mean, why do you think that it works?

- Because, to get this side [showing the back side (layer)]

CQ14. - 36

CQ15. - Can you show me how you did that?
- I counted these [showing the top layer] and counted again for this and this [showing the middle and bottom layers]

CH

CH is a fourth grader and is described by his teacher as a below average student in mathematics but very high in verbal ability. He said he saw similar cubes before but did not have similar toys except for Legos.

GQ1.
- 4 [just counted individual cubes without any hesitation]

GQ2.
- 8 [counted the cubes in rows starting from the top left and tapping each cube in the row twice while counting them one by one]

He was clearly seeing the building consisting of rows but he did not multiplied or skip counted the rows

GQ3.
- [Counted the columns of cubes starting from the far left. He tapped twice to each cube in the row to account for the cubes at the bottom] 12.
- Can you show me how you did that?
- 16 [he did the same thing but more carefully and this time he got it correct]
- How do you think that you missed some in the first trial?
- I counted the back but in the front I didn't count these [showing the front bottom row]

GQ4.
- 36 [counted the cubes in columns one by one by tapping each cube on the top three times]

GQ5.
- 12
- Can you tell me how you found?
- 5 + 3 = 8 and 8 + 4 = 12.

CQ1.
- 9 [counted the cubes]

CQ2.
- 8
- Can you tell me how you did that?
- Cuz, four on this side and four on this side [showing the vertical layers]

CQ3.
- 16
- Can you tell me how you found that?
- I saw four and four here [partitioning the front layer into fours] that is 8 and 8 on the other side, 16

CQ4.
- 36 [in order of columns, he counted each cube in the front three times to account for the cubes in the middle and back]
- Can you tell me about your strategy?
- I counted each one three times and found 36

SA

SA is a fourth grader and is described by his teacher as an average student in mathematics. She said she had never seen such cubes before. She played with the Legos a little bit in her childhood.

GQ1.
- 4 [counted the cubes in order of rows]
GQ2.
-8
-Can you tell me how you found it?
-1, 2, 3, 4 here [counting the cubes in the front layer one by one] and there supposed to be 4 in the back [implying the vertical back layer]
-So, you say 4 in the front and 4 in the back?
-Uh um.

GQ3.
-8
-8? Can you tell me how did you find it?
-Cause there is four here [indicating the top-back row] and four here [showing the top-front row]
-I want you to find out all of the cubes in the building
-16
-16?
-I counted the bottom.
-Can you exactly show me how you got that? How did you come up with this 16?
-There are four here [counting the bottom front row one by one] and there supposed to be 4 in the back
-You have two answers 8 and 16, which one do you think correct?
-16, because 8+8=16

GQ4.
-24
-Can you tell me about your strategy?
-Because, there is 9 on each side.
-Can you show me exactly? Can you show me the 9th cube first?
-There is 1, 2, 3... 12 [laughed after counting the cubes in the front layer one by one in order of columns] here and there is 12 on the other side [she did not realize the middle layer]
-How about the middle?
-[Looked at the picture for a while and stopped.
-Is this your final answer? Do you want to check it again?
-Yes, 24 altogether.

GQ5.
-16 [She didn’t take the numbers account she just tried to measure the space with the cube. She placed the cube 4 times on the front, 6 times on the right side and 6 times on the top]

CQ1.
-9 [counted the cubes one by one]

CQ2.
-8 [counted the cubes one bye one]
-Can you tell me how you got that?
-There is 4 cubes on each side [indicating the left and right vertical layers of 4 cubes]

CQ3.
-16
-I want you to tell me how you came up with this answer?
-There is 8 on each side [indicating the left and right vertical layers]

CQ4.
-[Looked overwhelmed, and counted the cubes in the front, turned it around halfway and counted] 28 [as she moved around it].
-There is 12 on this side and 12 on this side [showing the front and back vertical layers] and there is a bar up in the middle [showing the top middle row] it is 4 and the total is 28
-Can you show me exactly?
-12 here, 12 here, and there is an extra bar here.
-Do you think that this extra bar is not equal to the one on the side?
-I don’t know.
ST

ST is a fourth grader. Her teacher described her as an average student in mathematics. She said she didn't see the cubes before. She used similar ones not as building but loose cubes for counting and addition in the first grade. She also played with the Legos a little bit, but not much.

GQ1.
-4 [counted the individual cubes one by one]

GQ2.
-4 no I mean 8
-Can you explain me how you did that?
-There is 4 right here and 4 right here [indicating the bottom and top layer successively]

GQ3.
-9
-Can you tell me how you found?
-I counted up here [showing the top layer as she counted the cubes clearly in order of rows] and added one
-Can you show me where the 9th cube is?
-[She could not explain her answer clearly and she stopped].
-I want you to explain your answer.
-I counted 8 up there, 8 right here and 4 right here [she returned to a different strategy and counted the visible cube faces and found 19, making an addition error].

GQ4.
-there is 33 altogether [counting the cube faces at the outside]

GQ5.
-36
-Can you tell me about your strategy?
-We can put 3 here [front vertical edge] 3 here [left vertical edge] 5 here [right bottom edge] and 4 here [left bottom edge] and 3 here [right vertical edge] and a few here [for the top], 36

CQ1.
-9 [counted the cubes one by one in order of rows]

CQ2.
-8
-Can you tell me how you did that?
-1, 2, 3... 8 [counting the cubes in the top layer first and the two cubes at the bottom right and turned the building around and lost the track of her counting. She started over and found 9]
-Can you find the number of cubes without moving the building?
-1, 2, 3, 4 [this time she counted the cubes at the top layer and without moving the building] 5, 6, and 7, 8 [indicating the left and right bottom rows]

CQ3.
-[Counting the cubes all around the building] 17.
-Can you show me how you got that?
-1, 2... 8 [counting the cubes in the top layer] and 1, 2, 3, 4 here [counting the cubes in the bottom right row] and 1, 2, 3, 4 here [for the left row, after turning the building around], 8 [she forgot the first group of 8 cubes]
-So, you have two different answers, 8 and 17. Which one do you think is correct?
-8
-Can you show me the 8 cubes again?
-1, 2... 8 [for the top layer] 9, 10, 11, 12 [for the right bottom row] 13, 14, 15, 16, 17 wait, no 16 [she first double counted the cube at the far end of the building but realized that it must be 16. She became more systematic in her counting. She made two improvements in her enumeration. First, she organized
her counting based on rows of cubes. Second, she started to use counting on strategy instead of
counting local groups of cubes].

**CQ4.**
-32 [she tried to count the cubes at the outside but not counted the cubes in the back. First she counted
the cubes in the front, then at the top, left, and right side but not in the back. She didn’t account for the
cubes inside and backside]

**POSTINTERVIEWS**

**DA**

**GQ2.**
-8
-Can you tell me how you did that?
-4 on the top, 4 in the bottom, 8

**GQ3.**
-16
-Can you tell me how you did that?
-I counted 8 on the top and 8 in the bottom and 8 plus 8 is 16

**GQ4.**
-36
-Can you tell me how you found that?
-12 on the top, 12 and 12 [showing the middle and bottom layers]. Can I do like math problem?
-Sure
-[puts three 12's one below the other, adds them up and finds 36 as shown below]

![Image](image.png)

**CQ2.**
-8
-Can you tell me how you found that?
-4 on the top and 4 in the bottom

**CQ3.**
-16
-Can you tell me how you found that?
-4 here, 4 here [showing the top layer consisting of two groups of two rows] 8 and 8 in the bottom

**CQ4.**
-36
-Can you tell me how you found that?
-There is 6 here and 6 here [partitioning the top layer into two groups of 6 cubes] and it is 12, 12, and
12 more 36. [pointing to the middle and bottom layers]

-Can you tell me how you found that?
GQ2.
-8
  -Can you tell me how you found that?
  -There is two of these [showing the top layer of 4 cubes by circling around it]

GQ3.
-16
  -Can you tell me how you found that?
  -There is 8 and 8 [partitioning the building into two 8-cube buildings]

GQ4.
-36 [counting the cubes on the top layer touching three times to each cube to account the cubes in the middle and bottom layer]

CQ2.
-8
  -There is four and four [indicating the layers on the left and right side]

CQ3.
-16
  -There is 8 and 8 [partitioning the building into two 8-cube buildings]

CQ4.
-36 [counting the cubes on the top layer touching three times to each cube to account the cubes in the middle and bottom layer]

  how much is 4 times 9? 9, 18, 36

ST

GQ2.
-8
  -There is one and there is another one behind it so which makes two [showing the bottom left front-to-back row] and then one and another and there is altogether 4 [for the bottom layer] and then 4 right there [top layer] so it is 8.

GQ3.
-16
  -There is 4 and 4 and there is 4 and 4 [showing the front-to-back layers of 4 cubes] so 4, 8, 12, 16.

GQ4.
-24.
  -There is 6, 6, 6, and 6 [showing front-to-back vertical layers] 6, 12, 18, 24.
  -Can you tell me how did you find that exactly?
  -[Recounting the cubes in one layer], there is 9 [smiled] there is nine, there is 9, and there is 9. 9 plus 9 is 18 and 18 plus 18 is 36.

CQ2.
-8
  -4 plus 4 is 8 [showing the vertical layers]

CQ3.
-16
  -There is 4, 4, 4, and 4 [showing the vertical layers], is 16.

CQ4.
-36
  -there is nine [counted the cubes in one layer] and 9 plus 9 plus 9 plus 9 is 36.

SA

GQ2.
-8
   -There is 4 on this side and 4 on this side [showing the vertical layers]
GQ3.
-16
   -There is 8 on this side and 8 on the other side [showing the front and back layer]
GQ4.
-36
   -Can you tell me how did you find that number?
   -There is 12 up here [counting the cubes on the top layer one by one] 12 in the middle and 12 at the bottom [SA formed the new unit, layer and abstracted the equality of layers].
CQ2.
-8
   -There is 4 on each side
CQ3.
-16
   -There is 8 on each side.
CQ4.
-36
   -There is 12 here 12 here in the middle and 12 right here [showing the bottom layer]
INTERVENTION: SA&ST

Activity 1: A 2x2x2 building to be shared b/w 2 people

Int: You have this building and you want to share it equally between two people. How would you do that?

SA: There is 8 squares.

ST: Four for you [showing the top side] and four for me [showing the front side] and then two for you [showing the upper row on the right side] and two for me [showing the lower row on the right side. She was dealing with the cube faces]

SA: No, but there is 8 squares [meaning cubes] in whole. So each gets 4.

ST: There is 12 [counting the cube faces visible on the picture]

SA: They are gonna be sides.

ST: No, look this is two [touching the top sides of the two cubes in the top layer]

SA: They are just tops.

ST: [Looks at the drawing] Yeah [looked not convinced]

SA: See, there is two halves [She was functioning at the level of A conceptualization at least for this small building]. This half is 4. There is 8 squares [meaning cubes]. So, each gets 4, right?

ST: Uh um [still looked not convinced].

Int: Are you agreed on this?

ST: Look, look, there is two right here, and two right here [showing the faces visible on the top side of drawing]

SA: They are top of the square [meaning cube]

ST: There is two right here [showing the cube faces in the upper row of the front side]

SA: This is side, top, and side of square [showing the three faces of the cube on the corner]

ST: There is two right here [showing the upper row in the front side]

SA: This is the top of the square [meaning cube]. SA did not realize that ST started to see what she was explaining. She meant the upper front row. As if ST was starting to see the drawing as a 3-dimensional object as she was indicating the row consisting of two cubes.

Int: Let her explain what she means.

ST: See, there is two right here [again tracing the boundary of the cubes in the upper front row] and two right here [showing the bottom row in the front] and one here [pointing the cube at the bottom right end] and there is another you can't see.

SA: OK, you see this line dividing the two halves [showing the middle vertical line on the right side]. This is four squares [cubes] over here. This is the side of the square [showing one face of a cube].

Int: Cubes, let say cubes.

SA: Right, cubes. This is the side of the cube and this is the side of the other cube [explaining the picture]. This is one cube and 8 cubes in all. And we want to share right in the middle and each gets four.

Int: Do you agree with her [to ST]?

ST: Uh um.

Int: Can you shade the two equal parts in two different colors?

SA: What do you mean?

Int: Can you shade one share in one color and one share in another color?

ST: Can I just 4 here and 4 here?

Int: OK.

ST: So, I color my share and she colors her share?

Int: Yes.

ST: OK, I colored.
Int: I'll show you this building. This is the actual cube building. So, can you see the equal parts here [indicating the cube building]?  
Both: uh um.  
Int: How many each has?  
ST: 4  
SA: 4  
ST: Altogether 8 but each gets 4.  
Int: Can you build this building with cubes each of you giving equally?  
SA: Let's put 4 in the bottom.  
ST: uh um and four on the top.  
Int: Can you share it right now as you did on the drawing?  
ST: [shows as in the figure]  
SA: [shows the other way]  

Int: Can you do it in another way?  
SA: [makes the third way]  
Int: Good job. You guys are great. Let's move to the next problem.  
Activity 2: A 2x2x2 building to be shared b/w 4 people  
Int: I give you these pictures. You have this building. You want to share this building equally between 4 people. How would you do that?  
SA: There is 8 altogether. We get 2 each.  
ST: Yeah.
Int: Show me just one share only. I mean, shade one share on your picture.

SA: [shaded the face only]

Int: SA, can you shade the whole share. You said these are the faces. I want you to shade the whole share.

ST: Yeah, you need to shade the ones besides that [implying the other visible faces of the cubes]

SA: [Completed her shading immediately]

Int: OK, great. Can you make one share out of cubes?

ST: Yeah [puts two cubes together.

Int: Make another one. SA, you make another one and put them together [each makes one share]. Put them together. OK, we put two shares. Can you make one more share each? [Each of them made one share]. OK, put them together with the others and make the building. [They make the building]. Can you see any other way of sharing the building?

ST: Yeah, this is one part [as SA's shading]. Or, you can just go [pauses, confused]. I don't know.

Int: OK, you just show me the two ways, this and this [showing the ways they did before as in their shadings]. I think there is one more [waits a little bit, then] How about this one [showing the third way as in the figure].
Int: Let me repeat all the ways we did until now [repeats]. See we have three ways of sharing this building.

Activity 3: A 3x2x2 building to be shared b/w 3 people

Int: Let's take a look at this building [introduces the task sharing a 3x2x2 building between 3 people]. This is a slightly larger building, right?
Both: Yes.
Int: You have this building [showing the drawing of a 3x2x2 cube building]. You want to share this building equally between three people. Let's share the building first on the picture. Think about first, and shade only one share out of three.
SA: [Tries to find the number of cubes in the building by counting the cubes and replies] 12 square divided by three.
Int: Yes, but cubes.

PICTURE 5

ST: [Shaded a row of three cubes only, her solution is similar to the previous solution]
SA: [Shaded two adjacent cubes only]
Int: Can you check your answer? Think about. If you put three of them together, do they make the whole building?
SA: [Whispered to ST] three times four is 12 [while she was completing her shading]
ST: SA!
Int: You can tell her loudly, SA.
ST: She is shy. She doesn't speak a lot.
Int: SA, let's do it together. I want to learn how you do it, OK?
SA: [Just looks at and continues to shade her picture]
Int: ST, how many cubes in your share?
ST: 3
Int: OK, how many shares?
ST: 3
Int: OK, 3 times 3?
ST: 9
Int: Good, what do you think [indicating her picture]?
ST: I don't know.
Int: SA, how many cubes in one share?
SA: 4 [completed her shading but in an unexpected way. Since she first finds the number of cubes in the building and divides by the number of sharers she just picks any cubes to shade which are equal to what she found for one share]
Int: This is very smart. I didn't even think about this way. This is the building [showing the actual cube building]. Can you see that this is the same as the picture you have?
ST: Yeah.
Int: OK, if you share this building between three people, how would you do that?
SA: [Trying to find the total number of cubes by counting individual cubes]
ST: There is three on each side, three, three, and three [showing the three rows of three cubes]
Int: Total?
SA: 12
Int: 12, OK. ST, I want you divide the building not in groups of three cubes but in three groups.
SA: 3 divided by 12 [meaning 12 divided by 3]
ST: [Again] Three right here, three right here, and three right here [showing the rows of three cubes, and pauses, she just recognized that there are four of them]
SA: Each of us gets 4 squares [cubes]
Int: You have two different answers. ST, your thinking is good but you have 4 shares. It should be three. So, I want you to make three equal parts.
ST: I don't get that.
Int: See you are counting three cubes here [showing the row]. How many of these three cubes are there in this building?
ST: 4
Int: OK, I want you to make it 3. Three groups of cubes, not groups of three cubes.
SA: [looks impatient and knocks her head]
Int: Let's put the building this way [turns the long side up]
ST: Three here, three here, three here, and three here [showing the columns of three cubes]
SA: [discovers another way, takes the pen and shows the three layers of 4 cubes one by one, but ST did not recognize it]
Int: Can you show us how you did it [to SA]?
SA: 3 people. You have three extra squares [cubes]. First square [cube] for this [showing one column] second square [cube] for this [showing another column] and one square [cube] for this.
Int: this is something different. Can you do it again?
ST: [interferes] He wants you to do this [showing the share in her picture].
SA: Actually, I did these four, these four, and these four [showing the groups of four cubes on the building as she did in her shading]
Int: Let me give you cubes and you make this building out of cubes. But first make one share each.
Can you show me your share [to SA, after she made the building]?
SA: These 4 [showing the groups of four cubes as in her shading]
Int: Can you pull it apart?
SA: [She holds a vertical layer and pulls it apart then divides the other two layers in the same direction]

PICTURE 7

Int: Is it different from yours [to ST]?  
ST: Yeah.  
Int: Can you show me your solution on the [concrete] building?  
ST: Yeah, but I don't have the same [paused]  
Int: OK, you can change your answer now. I can give you another paper to shade the share. Here you go. Can you make it again?

PICTURE 8

Int: Is that OK? What do you think SA?  
SA: It's OK.  
Int: Now, each of you has different solutions and both are correct. If I tell you that there are still other ways. Can you think about it a little bit?  
SA: [Showed her way again]  
ST: [Showed her way again]  
Int: How about this [shows another way of partitioning the building into three]? That's enough. Let's take a look at the next problem.

Activity 4: A 3x2x2 building to be shared b/w 4 people.

Int: I will give you these loose cubes and you will make the building by four people each of you giving equally.  
ST: [struggling to make the building with shares similar to what SA made for the previous activity]  
SA: [made the building (a) and said] They each gets 3 cubes [She turned the building up and pulled one share apart (b)]
ST: [She managed to partition the building into 4 equal parts. She might just have copied what SA did (c)]

Activity 5: A 2x2x4 building to be shared b/w 4 people.

Int: Good job. Let's take one step further. This time you have this building [giving them the drawing of a 2x2x4 building] and you want to share it between 4 people.
ST: [again, shaded 3 cubes only as in the previous activity]
SA: [Shaded one column of 4 cubes already, with a little mistake. She forgot to shade the top]
Int: You have different solutions. ST can you explain your answer how 4 people get the equal amounts?
ST: [She was just trying to imitate the solutions of previous activity although the building got larger].

Int: OK, here is the actual cube building. ST can you show me the 4 equal amounts on this building?
ST: Right here, right here, right here, and right here [showing the 4 columns]
Int: OK, great. So, do you need any correction on your picture?
ST: Yeah [completes the shading to make it four cubes]
Int: Good job. Here are some cubes. I want each of you put one share only.
ST: [Again, built a 3-cube column and realized that one is missing. She put the 4th one on top of the others]
Int: Perfect. So, put them together and make the other two shares, each of you one share.
Both: [made their shares correctly and put them into the building]
Int: Can you think about the other ways of making this building?
SA: [made two different ways of partitioning the building (a and b)]
ST: [made another two ways (c and d)]

Activity 6: A 4x3x2 building to be shared b/w 3 people.

Int: We have a larger building here [showing the drawing of a 4x3x2 building] Three people want to share this building equally.
SA: [shaded 4 cubes, half the front layer]
ST: [shaded 4 other cubes, 2 thirds of a vertical layer]

Int: There are three people and you need to make three equal shares.
SA: [Started to shade the other four cubes in the layer. She correctly shaded one third of the building]
ST: I got it [shaded the remaining two cubes in the layer. She had shaded one fourth of the building].
Int: OK, ST let's think about your picture. You have this slice and three people. When you repeat this slice 1, 2, 3, 4 [as he touched on the vertical layers with the pen]

ST: OK, there is 4 on, wait. There is six on this side, six on this side [showing the right and left vertical layers] and six in the middle [she did not recognized that there are two layers in the middle]

Int: Six in the middle?

ST: Yes.

Int: How about this one [showing the other vertical layer in the middle]?

ST: 1,2,3,4,5,6, [counting the cubes in the top middle two rows] {She was overwhelmed with the increasing size of the building and she could not visualize the middle part and she was able to attend only the outside of the building, a C type thinking}

Int: How about that one [showing the middle-bottom-horizontal layer (d)]

ST: No. I am saying, oh yeah, [laughing]

Int: Think about it. I can give you another picture to shade.

SA: Can I get another picture?

ST: I don't get it.

Int: You don't get it? [gives another picture]

SA: Do we shade one share for each person?

Int: Yes, shade one share only.

ST: [waiting while staring at the picture]

Int: Do you have any question?
ST: No.
Int: Let me explain this one [showing her previous shading task]. You shaded this one [a six cube layer]. This is good. If we were 4 people instead of 3, it would be OK. See the four slices here, 1, 2, 3, and 4. But we are 3 people. So, you need to think about it and make three equal parts.

ST: So, OK. I have like there right here [showing the top-back-horizontal row] that would be the outside [tracing on the back vertical layer] and then these on the bottom [showing a row from the bottom].
Int: Yes, it might work. Can you show me again?
ST: OK, I don't know how much right there [showing the top]
Int: You don't know how much up there?
ST: No.
Int: Do you know how much right here [showing the front vertical layer]?
ST: 8 [counting the cubes one by one]
Int: OK, 8. Do you think that this 8 is equal to this and this [showing the middle and back vertical left-to-right layers] because these are right behind it?
ST: Yeah.
Int: How many eight's?
ST: Three.
Int: OK, can you shade one of them?

PICTURE 15

Int: OK, this is the actual cube building. I want you to show me what each of you shaded.
Both: [showed what they shaded individually]
Int: Do you think that there is any other way to share this building?
Both: [paused]
Int: let's make the building out of cubes. Make your shares and put them together.
Both: [made their shares and put them together]
SA: There supposed to be a bar at the middle.
Int: let me do the last share.

Activity 7: A 4x3x3 building to be shared b/w 3 people.

Int: [reads the question]
ST: [immediately shades the three horizontal layers]
SA: [tries to partition the building in an unusual way, but she fails. She finds a number in her head and tries to shade that number of cubes on the drawing]
Int: ST, how many cubes in each share?
ST: 9
Int: Can you show me?
ST: 12 [counts the cubes in the top layer one by one]
Int: How many altogether?
ST: 30
Int: How did you find that?
ST: 12 plus 12 plus 6 equals 30.
Int: Why six? If they are all equal?
ST: [pauses]
SA: [waiting ready to say something and calculating something with her fingers]
ST: 24 plus 12 is 41 [counted the cubes on the top layer, touched the cube at the front left middle row, and front left bottom to account for the middle and bottom layers respectively. However, she made an addition error and said 24 plus 12 is 41]
SA: No.
ST: 36 [immediately corrected herself]
Int: Can you show me how you find that?
SA: Well, you add 24 plus 12.
Int: Why is it 24 plus 12?
SA: That was fair.
Int: right, we have three shares [showing the three layers] and if there is 12 in each, then?
ST: yes, 36.
Int: What do you think SA? Your picture is different.
SA: I found 24 cubes in the building and I divided 24 by 3 then each gets 8. [She was relying more on numerical solutions. If she gets the total number of cubes correct then she can solve the partitioning correct]
Int: Let me show you the real building. Can you check the building and the picture if they are the same?
Both: uh um [nodding their heads]
Int: OK, how do you know that they are same?
ST: Because, there is 12 up here [showing the top layer] and three rows [indicating the middle and bottom layers].
Int: OK, what do you think SA?
SA: 24 cubes [counted the outside cubes visible on the top, front, right side and the back but not at the bottom] here and 24 cubes here [showing the 4x3x3 cube building]
Int: OK. Can you make one share with cubes? Just one share each OK?
SA: [made a 4x2 vertical layer, as in her shading task]
ST: [made a 4x2 horizontal layer, looked at it for a while and then added 4 more cubes to make it 4x3 as in the bottom layer of the building]
SA: [completed another 4x2 vertical layer and put them together to make 4x2x2 building]
Int: Let’s make the other shares and make the whole building.
ST: [started to the second layer of 4x3, held it up and put on top of the other. Now, her building was 4x3x2]
SA: [made another 4x2 vertical layer and put it next to the other two verticals. Her building now is 4x3x2 instead of 4x3x3]
ST: [started to make the third layer. This time she started to build the layer on top of the building adding one or two cubes at a time] {She has no difficulty visualizing the concrete building}
SA: recognizing that the building are not same, she started to add another layer to her building putting cubes one by one on top of the building} {she was not systematic in her construction of the whole}
Int: OK, you seem ready. How many do you think you have in your buildings? SA?
SA: 36 [after a while]
Int: OK, now you have 24 and 36. Which one do you think is correct?
SA: 36 [counted the front layer and thought a little bit more as if she calculated something in her head]
Int: Can you show me how you did it? Or, how can we be sure that there are 36?
SA: [Started to count the cubes around the building]
Int: Let me show you something [interrupting her counting]. How about this [separating a 3x3 vertical layer from the building]
SA: 9
Int: How many of these [showing the 3x3 vertical layer] go into this building?
SA: 4
Int: Do you know how much is 4 times 9?
SA: 36
Int: What do you think ST? How do you know that there are 36 cubes in the building?
ST: Because 12 blocks here [showing the top layer] and 12 right here [middle layer] and 12 block right here [showing the bottom layer] and altogether 36.
Int: OK, can you [to SA] check your picture if you colored the share correct? I can give you a new picture [gives a new picture to SA]
SA: [Started to color the three horizontal layers (a)]
ST: [With a little help from the interviewer, ST invented another way of sharing the building between 3 people (b) while SA was completing her coloring].
Int: OK, SA how many red cubes do you have [in the top layer]?  
SA: 12 [counting the cubes with her eye movements]  
Int: How many blue cubes do you have [middle layer]?  
SA: [stops for some reason]  
Int: If they are equal shares, what do you think?  
SA: [waits for a while]  
Int: Can you make the red layer with cubes?  
SA: What do you mean?  
Int: Can you show me the red cubes in the building?  
SA: [took the top layer apart from the building]  
Int: OK, put this aside and show me the blue layer?  
SA: [took the middle layer apart from the building]  
Int: Can you show me the green part?  
SA: [took the last layer and put it on top of the others and that way she re-constructed the building with three layers]  
Int: How many green cubes do you have?  
SA: 12  
Int: How many blue cubes do you have?  
SA: 12  
Int: How many red do you have?  
SA: 12  
Int: How many altogether?  
SA: 36  

Activity 8: A 4x3x3 building to be shared b/w 4 people

Int: We are 4 people and we want to share this building [showing a 4x3x3 building]  
ST: Can I pick my color?  
Int: Yes, pick your favorite color and show me one share.  
ST: [after she colored one] can I make the others?  
Int: Let's wait and discuss a little bit.
SA: Can I get another picture? [She thought that she messed up with the first one (b)]
Int: Sure, take this one
SA: [looked confused, and jot down on the picture]
Int: You have different shapes, different shares. ST how many do you have in one share?
ST: 9
Int: SA how many do you have?
SA: I didn't color.
Int: OK, can you do it now?
SA: 6, six in each share.
Int: OK, four people, how many altogether?
SA: What do you mean?
Int: If there are 6 in each share and if there are four people, how many altogether?
SA: 24
Int: Let's make the buildings. Make one share with cubes. One share only.
ST: Shall I make mine?
Int: Yes, do it yours.
ST: [takes a 3x3 layer and puts it aside]
SA: [takes a 3x2 piece apart from the building as in her coloring task]
Int: So, you find your share in the building, right? How many of these [showing her 3x2 unit] do you think are there in this building [showing the whole building]?
SA: [Shakes her shoulder to mean I don't know]
Int: What do you think, ST? How many of these [showing her 3x3 unit] are there in this building?
ST: 36 [misunderstood the question]
Int: If we count this one [the 3x3 unit] how many of them in this building?
ST: There is 9 here [showing the colored share in the picture] and 9 here [showing the 3x3 layer made of cubes]
Int: Ok, this is one [showing the 3x3 layer] and how many of these are there in the whole building?
ST: There is 4.
Int: Yes, because there are 4 people and each gets this much [showing a layer]. Let's make the buildings. Put equal parts together and make the buildings.
ST: [carries the 3x3 layers from one building to another]
SA: [puts two 3x2 layers together]
Int: SA, can you make the third share.
ST: Can I?
Int: Yes.
SA: [now has a 3x3x2 building]
ST: [now has a 4x3x2 building]
Int: Ok, we have different buildings. What do you think?
Int: Let's return to the picture and color the different shares.
SA: [seemed to be tired]
Int: You want to take a break?
SA: [no answer, seemed not to be concentrated on the task]
Int: Let's do another one. [Introduced a 2x2x4 building, long side up]. Let's share this building between 4 people. SA, choose your favorite color.

Int: You have two different ways of sharing this building between 4 people. Can you show me how did you do it?
ST: Four right here, 4, 4, and 4, 1,2,3,4 [showing the 2x2 horizontal layers]
Int: SA, what do you think? Can you show me your solution?
SA: [shows 1x4 columns one by one as she moves her finger vertically along the columns]
Int: Let's make the shares with cubes exactly the way you colored them and let's see if we get the same buildings.
ST: [builds a 2x2 horizontal layer]
SA: [builds a 1x4 column]
Int: Your way and your way are different. Let's make the other shares and make the buildings. Both: [made the buildings correctly]
Int: Can you pull just one share apart?
Both: [did the separation correctly]
Int: See, are they equal?
Both: Yes.
Int: Let's go over this one again [showing the drawing of the 4x3x3 building]. We have this building and we want to share it between 3 people.
SA: [silently shaded the front vertical layer as one share]
Int: Great, you did a good job. Give me five.
SA: [gives a five to the int.]

Activity 9: A 4x4x3 building to be shared b/w 3 people
Int: This is the last activity I want you to do. I know you got tired but let's finish it quickly. We have this building and we want to share it equally between 3 people. Can you show me the three shares in three different colors?

![Building A](image1.png) ![Building B](image2.png)

Int: You guys are great. Can I ask you a question, SA? How many cubes are there in one share? For example, how many gray cubes?
SA: 16
Int: [just to make sure they internalized the behavior] Let's make the building with cubes. Make the shares separately and put them together if we get the same building.
SA: [builds the 4x4 first layer and continues to build it up]
ST: [builds the 4x4 first layer and continues to build it up. As in the other tasks, she first forgot to put the upper layer and then she realized that she should add one more layer]
Int: SA, how about if we were to share this building between 4 people instead of three?
SA: [separates the building into 4 vertical left-to-right layers]
ST: [divides the building into 4 vertical front-to-back layers]
Int" Can you show me another way of sharing this building between 3 people?
SA: [made two 4x2x2 and one 4x4x1 building]

**INTERVENTION: CH&DA**

**Activity 1: A 2x2x2 building to be shared b/w 2 people**

Int: Let's read the first question. You have this building [showing the 2x2x2 building] and you want to share it equally between two people. How would you do that? Do you understand the question?
CH: Yeah. You want me to do?
Int: Please.
CH: I give one person this one [showing the 2x2x1 vertical layer] and one person this one [showing the other layer]
Int: DA your solution is same?
DA: Yeah, same.
Int: OK, I want you to shade one share with a color, just one share. Leave the other share uncolored.

CH: [Made the coloring very quickly as in figure below]

DA: [Colored just the front face first then asked if he needed to color the sides]
Int: See the question is two people sharing this building, so shade only one share. You said 4 cubes for each person, right? Can you shade the four cubes only?
DA: [Looked puzzled and hesitated]
Int: You shaded only the face not the whole cubes.
DA: Do I shade the whole? [Started to shade the entire building]
CH: No, not the whole thing. Look, you have to do that and you have to do that [showing the top and the right side of the front layer in DA’s drawing]
DA: [Colored the faces of front layer, with no indication of understanding]
Int: Do you know why? Why we shaded this and this, too [showing the top and the right side of the front layer]
DA: To make it equal.
Int: That’s right, to make it equal. What else?
CH: So you can see the 3D in it.
Int: OK, because it is 3D.
CH: So you can see altogether in it.
Int: DA do you know what 3D is?
DA: Yeah, like, like, like. It’s like pups up [He doesn’t seem to have the understanding that a 3D occupies a space rather he thinks that a 3D consists of an organization of individual faces].
Int: OK, can you see the 3D in this picture [showing the picture he already colored]
DA: No.
CH: I can.
Int: Can you the 3D in this [showing an uncolored drawing of a 2x2x2 cube building]?
DA: Yeah.
Int: You couldn’t see in this because you colored it too dark [joking]. Let’s make the building with cubes. You make one share you make one share. Put them together and make the building.
Both: [each made one share of a 2x2x1 vertical layer out of cubes]
Int: DA can you see the 3D right now?
CH: Yes. This is 3D [showing the cubes]. He is 3D. You are 3D.
Int: OK, let’s put them together. Now, think about how many ways of sharing this building between 2 people you can find.
CH: Yes [makes two different ways of vertical layering].
DA: [Separates the top and bottom layers first, then left and right layers and thinks about for a while and makes the front and back vertical layering with concrete cubes. He doesn't have any difficulties with small concrete buildings but pictorial and/or large buildings as evidenced in his enumeration tasks]

PICTURE 22

Int: Let's move to the next question.

Activity 2: A 2x2x2 building to be shared b/w 4 people

Int: You are a team of 4 people and you want to make this building each of you giving equally. I want you to show me one share first on the picture.
DA: He puts four and I put four.
CH: No [inaudible sound]
Int: Listen, go ahead [to DA].
DA: He puts 4 and I put 4 on top of it and we make the building.
Int: But this time there are 4 people.
DA: OK, one, each person gets one, wait [calculating something with his fingers]. OK, they get each, no, they'll get two.
Int: Let's shade one share, OK? Pick your color.
DA: We shade this one [showing the right vertical layer]
Int: You said each gets two, right? So you shade two cubes only.
CH: [already shaded one share, two cubes]
DA: [Shaded two cubes exactly the same way CH did]
Int: Ok, shade the other shares in different colors.
Both: [shaded correctly but exactly the same way]
Int: Let's make the building with cubes.
DA: [made a 4-cube bottom layer]
CH: [completed the building]
Int: DA, show me the yellow cubes in the building.
DA: Right here [holding the correct cubes]
Int: Put it aside and show me the green ones.
DA: Right here [touching on the correct cubes]
Int: OK, let's move.

Activity 3: A 3x2x2 building to be shared b/w 3 people

Int: [Introduces the third task]. Read the question. I want you to think first, decide on the share, take a color pen and shade it.
Both: [Shade the vertical layers correctly]
Int: Can you shade the other shares in different colors?
Both: [shaded correctly]

![Picture 23]

Activity 4: A 3x2x2 building to be shared b/w 4 people

Int: Let's look at the next problem. Can you read the question?
CH: I'm thinking.
DA: OK, there is 4 people.
Int: Right, and we want to divide this building into 4 equal parts.
CH: Everyone should get the same?
Int: Yes, they should.
DA: [started to make the building with cubes]
Int: CH, you can use the cubes too. But if you can do with the pen, then do it first.
CH: [goes over and over the vertical layers one by one while whispering something and suddenly raises his hands and says] Oo, oooo.
Int: You got it?
CH: Yes.
Int: OK, shade it.
DA: [made the building with cubes same as the picture and partitioned it into 4 groups and said] I see, each person gets 3. [He started to see the cubes in the picture but still he cannot partition the shares on the picture]
Int: Ok, put them together.
CH: [Completed his shading task]
DA: [Puts a row of 3 cube aside, another row beside it to make the bottom layer and made the top layer and started to shade his picture]

![Picture A](image1)

![Picture B](image2)

**PICTURE 25**

Int: OK, they look like the same. Let's think about them. This is the building, right [showing the 3x2x2 cube building]? They are same, right [showing another building]? Let me put this way [placed the building long side up]. How many people?
Both: 4
Int: OK, we want to share this building between 4 people.
DA: Same. Each gets 3 [seemed to realize the 4 column structure].
CH: [Takes the concrete building and explains the four shares on it as he did on his shading] You take this, like this, like this [three L shape figures], and like this [the remaining back row]
Int: Ok, I got it. Take this new picture and each of you shade what you already did.
DA: Do we have to do the same thing, like this [showing the vertical layers as in the previous task]?
Int: No, you showed me something on the building, right? Make it the same way. Four equal parts [tauching on top of a column on the building].
DA: Like this, like this, like this and this [showing the 4 columns]
Int: Right, shade one of them.
DA: [Started to shade one of the vertical layers again]
Int: No, no. Let's think about it on the [concrete] building. One share is this one, right [showing one column]?
DA: Yeah.
Int: OK, how about the others?
DA: [Turns the picture to orient it with the building] This one, right [showing the column on the concrete building]? So, I'm gonna shade this one [showing one of the columns in the picture].
CH: colored his picture in the same way but this time he tried to rotate the building by 90 degrees and made coloring error

Int: CH, you did the sharing in an interesting way. Can you explain it to us?
CH: shows exactly what he did in his shading

DA: [Also made a face error but he corrected after the interviewer reminded him. He asked] Shall I do this [after shading each column]? [Still hesitates while shading the right side of the building]

Int: Yes, this is the building, OK?

DA: Uh, OK. [Completes the right face of the front-upper row. Takes another color and colors the two cube faces on the right side at the bottom] [he cannot coordinate the two views of the same cube]

Int: Is it different color?

DA: Yeah.

Int: Can you show me how you did it [on the concrete building]?

DA: This and this goes like this [showing two columns in the 2x2x3 building].

Int: OK, can you show me the blue part?

DA: [Shows correctly].

Int: Can you show me the red part? This one [showing the two cubes together]?

DA: No, right here [realized the mistake in his shading task]. Can I change it?

Int: Sure, change it.

DA: [completes his shading task correctly].

Int: Ch, see this is different [showing DA's shading] from yours, right? Can you show me how DA did this with cubes?

CH: Yeah. I think so. DA took this [showing a column in the concrete 2x2x3 cube building] he took this [showing another column] and he took this and this [showing the other columns].

Int: DA, can you show me how CH did it?

DA: [built the building with cubes after looking at the picture for a while] He did like this [separating the building into 3 vertical layers].

Int: He did something more, right? Put them back.

DA: [Playing with cubes] it has to be three in each. [numerically he knows what it should be but he cannot do the task spatially]

Int: Ok, let's put them away.

Activity 4: A 2x2x4 building to be shared b/w 4 people

Int: Let's take a look at this one. Read the question.

CH: This is easy [attempts to take the cubes to build the building].

DA: [tries to make the building with cubes]

Int: No cubes, no building, just use the picture and color pen.

CH: This is one, two, three, and four [showing the four columns].

DA: There is four, four, four, and four [showing the four horizontal layers].

Int: OK, I'm giving you four colors, shade the building.
Int: OK, yours is different, and yours is different. Can you make the buildings?
Both: [made the buildings]
Int: CH, can you show me the four equal parts in the building?
CH: [shows 4 columns touching one by one].
Int: DA, can you show me the four equal parts in your building?
DA: [Shows 4 layers by holding them one by one]
Int: Did you notice the difference?
Both: Yeah.

Activity 5: A 4x3x2 building to be shared b/w 3 people

Int: Here is another problem. We have this building and we want to share it between three people.
   Three people, three equal parts.
DA: Ok, three people get these three [touching three of the four front-to-back rows] and share this
   [the fourth row] {seemed to be confused and not sure}
CH: This, this, and this [showing the vertical left-to-right layers.
Int: OK, here is a building. Are they same [concrete and drawing]?
DA: Yes.
Int: How do you know? How can you make sure that they are equal?
DA: There are three here, three, three, three [showing the top rows].
Int: CH, what do you think?
CH: [Shows the vertical left-to-right layers].
Int: Let's make the buildings. Can you make your building [to CH]? And Can you make your
   building [to DA] with cubes?
DA: [Made a 3x3x2 building instead of a 4x3x2 building]
Int: Are they equal?
CH: No.
DA: [Adds another layer to make the building 4x3x2 and partitions the building into 4 vertical
   layers then takes the fourth one and distributes over the other 3].
CH: [seemed to be bored. Partitions the building into a number, which is not related to the
   question and laughs].
Int: Ok, DA, make the building again. Can you do it in another way?
DA: Yeah. Like this [slices the building into 3 equal parts, organized into left-to-right vertical layers].
Int: CH, can you do it in another way?
CH: Yeah [divides the building into three layers].
Int: OK, great. I want you to shade each share in different color on the picture the same way you did it now.
Both: [started wrong by mistake but they realized that they colored wrong and they corrected by themselves].
Int: You should read the question and decide on a solution first and then make the shading.

PICTURE 28

Activity 6: A 4x3x2 building to be shared b/w 4 people

Int: Let's do the next problem. This time we have the same building but 4 people want to share.
CH: That's easy.
DA: [after reading the question] 4 people, hu? [made the building with cubes and started to distribute rows of cubes to four people and distributed extra rows to four people. He got 4 loose piles of 6 cubes].
Int: So, how many each gets?
Int: How many altogether?
DA: 24 [counts the cubes one by one till 24] altogether 24.
CH: [made 6 cubes of 4 vertical layers].
Int: DA, can you put them together?
DA: [could not put them right away because they were not in order but he made the building, anyway].
CH: See, I am doing something different.
Int: Ok, CH, can you shade just this share [indicating the 3x1x2 layer] in the picture?
Int: DA, can you divide the building [concrete] into 4 by pulling them a little apart?
DA: Like this [divides the building into 4 vertical layers]?
Int: Ok, I want you to shade only this one [showing the second layer from the left] in your picture.
Both: [completed shading one share]
Int: DA, let's check how many of these are in this building [showing the layer he already shaded and the building itself].
DA: 1, 2, 3, 4, 5, 6 [counting the cubes along the row in the shaded layer] ooo! I colored wrong way [showing the front-to-back vertical layers].
Int: Ok, how about you, CH?
DA: Ooo! He made the same thing [meaning error] {In fact, DA could not understand how CH divided the building into four equal parts}.
Int: OK, I want you to shade the four shares into four different colors.
CH: Don't copy [to DA].
DA: I'll do this way. Ooo, it's hard.
CH: No, it is not.
Int: Let's do this way [sliced the building into four equal layers]. Look at the building [to DA].
DA: So, it goes like that, that [showing the layers on the picture].
Int: Right. Shade one share only, then second, third, and fourth.

Int: CH, are they different?
CH: Yeah.
Int: Show me how different.
CH: He did like this [showing the front-to-back vertical layers in the picture one by one].
Int: Can you show me how you did it?
Activity 7: A 4x3x3 building to be shared b/w 3 people

Int: You are free to choose your favorite color.
CH: This is easy. We did that before.
DA: Yeah. We did that before.
Int: No, you didn't.
Both: Yes we did.
Int: I know you did something else. See? You did sharing 6 this is 7. The task is similar but not the same.
DA: I see.
CH: Between three people?
Int: Think about first, mentally solve and then color it.
DA: I know. See, these, these, these, and these [showing front-to-back vertical layers].
Int: DA, if you do it like that how many shares would you have?
DA: 4. Oh. I see. This way, this way. See? One, two, three [showing the three left-to-right vertical layers].
Int: Right.
DA: So, do I have to shade this way [showing the three layers again].
Int: CH?
CH: Oh, I got it [started to shade the middle horizontal layer].

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Int: DA, you have three layers, right?
DA: Uh um.
Int: How many cubes in one layer?
DA: 12 [counting the cubes with eye movements].
Int: Ok, how many do you think altogether?
DA: I think, [waits] I can do it like a math problem.
CH: Can I say? Can I say?
DA: [wrote 3 12's one below the other and added them up, found 36].
Int: DA, take this pen and do it. CH, you can do it right here [showing his paper].
CH: 40 [mentally calculated].
Int: You found 40?
DA: 36 [added three 12's]
Int: 36, OK. CH?
CH: 38
Int: OK, let's think about it. How many blue cubes do you have?
CH: 12
Int: How many red cubes?
CH: 12
Int: How many yellow?
CH: 12
Int: How many altogether?
CH: 12, 24, 38 [making a counting error].
DA: No, you going…
CH: OK, 36
DA: 12, 12, 12 do math problem [to CH]
CH: 36
Int: Can you explain how you did it?
DA: I just counted this [front vertical layer] and it is 12 and I just added two more and I got a math problem [meaning 12 plus 12 plus 12 equals 36]
Int: Before it was not math?
DA: No, I just counted.

Activity 8: A 4x3x3 building to be shared b/w 4 people

Int: This time we have the same building but we want to share it between 4 people. Think first, do it mentally and pick your favorite color and shade the 4 equal shares.
CH: OK, that's easy.
DA: I got it.
Int: [While they are coloring the shares] Can I ask you something? Are you good at multiplication table?
CH: Yes.
DA: You mean times?
Int: Yes, 5 times 7?
DA: I know it, 35.
CH: 35 [mentally added fives]
CH: Do you know 12 times 12?
Int: Too much.
Both: 144 [together].
PICTURE 32

Int: [After they finished coloring] I have a question for you CH. How many shares do you have, now?
CH: 4
Int: How many cubes in each?
CH: There are, um, 9.
Int: OK, how many altogether?
CH: 36
Int: How do you know?
CH: I added all up.
DA: [Put four 9's one below the other and added them to find 36]

Activity 9: A 4x4x3 building to be shared b/w 3 people

Int: [Introduced the question].
DA: [after reading the question] Look at this, watch, watch. There is right here, right here, and right here [showing the three horizontal layers]
CH: How come, this is what I see, like that, like that, and like that [going about the layers]
Int: OK, let's do the shading.

![Building A](image1)

![Building B](image2)

PICTURE 33

Int: How many cubes do you think altogether?
DA: [Counted the cubes at the top layer and] OK, there is three, right? [Wrote down 3 16's one below the other added them up and found 48].

Activity 10: A 4x4x3 building to be shared b/w 3 people

Int: Let's do this one. We are four people now.
Both: [Just did it without any hesitation].